LOCAL LEFT NOETHERIAN IPLI-RINGS

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All rings are associative and unitary. A ring R is a *pli-ring* (resp. ipli-ring) if every left ideal (resp. two-sided ideal) of R is of the form Ra for some $a \in R$. Clearly, every pli-ring is a left Noetherian ipli-ring. A ring R is called *local* if R has a unique maximal left ideal.

This note contains statements of some results concerning ideals and global dimensions of local left Noetherian ipli-rings.

A few definitions are needed. Let I be an ideal (i.e., two-sided ideal) of a ring R. We shall give two definitions of transfinite powers of I. The first is: $I^1 = I$; $I^\alpha = I \cdot I^\beta$ if $\alpha = \beta + 1$; $I^\alpha = \bigcap_{\beta < \alpha} I^\beta$ if α is a limit ordinal. The second definition is notationally distinguished from the first by writing the index ordinal in a square bracket; it goes as follows:

$$I^{[\omega^0]} = I; \qquad I^{[\omega^\alpha]} = \bigcap_{n=1}^{\infty} (I^{[\omega^\beta]})^n \quad \text{if } \alpha = \beta + 1; \qquad I^{[\omega^\alpha]} = \bigcap_{\beta < \alpha} I^{[\omega^\beta]}$$

if α is a limit ordinal. Note that the second definition defines transfinite powers only for ordinals of the form ω^α. For all the set-theory involved, we refer to [3].

The following theorem is basic.

THEOREM 1. *Let A be a proper prime ideal in a prime left Noetherian ipli-ring R. Then there exists an ordinal α such that $A^{[\omega^\alpha]} = (0)$. Let α be the first such ordinal. Then $A^{[\omega^\beta]} \subsetneqq A^{[\omega^\gamma]}$ if $\gamma < \beta \leqq \alpha$. The prime ideals of R contained in A are precisely those of the form $A^{[\omega^\beta]}$ where $\beta \leqq \alpha$.*

Recall that a *domain* is a (not necessarily commutative) ring without zero-divisors.

THEOREM 2. *Let R be a local semiprime left Noetherian ipli-ring with Jacobson radical J. Then*

(1) *R is a pli-domain.*

(2) *There exists an ordinal α such that $J^{[\omega^\alpha]} = (0)$. Let α be the first such ordinal. For every $\beta < \alpha$, choose $x_\beta \in R$ such that $J^{[\omega^\beta]} = Rx_\beta$.*

(3) *Every nonzero element r of R can be uniquely expressed as*

$$r = u x_{\beta_1}^{m_1} \cdots x_{\beta_s}^{m_s},$$

where s is a nonnegative integer, $m_i \in Z^+$, $\beta_1 < \cdots < \beta_s \leqq \alpha$ and u is a unit in R.

(4) *Every left ideal of R is two-sided. Ideals form a well-ordered set under reverse inclusion. Every nonzero ideal A can be uniquely written as*

$$A = (Rx_{\beta_1})^{m_1} \cdots (Rx_{\beta_s})^{m_s},$$

where s is a nonnegative integer, $m_i \in Z^+$ and $\beta_1 < \cdots < \beta_s \leqq \alpha$. Further,

$$A \mapsto \omega^{\beta_s} m_s + \cdots + \omega^{\beta_1} m_1$$

is an order-isomorphism of the well-ordered set of all nonzero ideals of R under reverse inclusion and the set of all ordinals $<\omega^\alpha$. The same bijection is an anti-isomorphism of the monoid of all nonzero ideals of R under the usual multiplication and the set of all ordinals $<\omega^\alpha$ under the usual addition.

(5) *Every ideal of R is of the form J^λ for some uniquely determined $\lambda \leqq \omega^\alpha$. If the Cantor normal form of λ is*

$$\lambda = \omega^{\beta_s} m_s + \cdots + \omega^{\beta_1} m_1$$

then $J^\lambda = (Rx_{\beta_1})^{m_1} \cdots (Rx_{\beta_s})^{m_s}$. In particular, $J^{\omega^\beta} = J^{[\omega^\beta]}$ for every $\beta \leqq \alpha$.

Recall that a ring R is *completely primary* if $R/P(R)$ is a domain, where $P(R)$ is the prime radical of R.

THEOREM 3. *Let R be a local left Noetherian ipli-ring. Then R is a completely primary pli-ring, every left ideal of R is a two-sided ideal and the set of ideals of R is a well-ordered set under reverse inclusion.*

R is a domain if and only if the well-ordered set of nonzero ideals of R under reverse inclusion is order-isomorphic with the set of all ordinals $<\omega^\alpha$ for some ordinal α.

More details about these rings (similar to Theorem 2) are obtained.

THEOREM 4. *Let A be a proper prime ideal in a prime left Noetherian ipli-ring R and let λ be an ordinal such that $A^{[\omega^\lambda]} \neq (0)$. Then there exists a set of principal right ideals of R which form a well-ordered set under inclusion order-isomorphic with the set of all ordinals $<\lambda$.*

Theorem 4 and a theorem of B. L. Osofsky [2] imply the following.

THEOREM 5. *Let A be a proper prime ideal in a left Noetherian iplidomain R and let λ be an ordinal such that $A^{[\omega^\lambda]} \neq (0)$. Then*

$$\begin{aligned} \text{r. gl. dim } R &= \infty && \text{if card } \lambda \geqq \aleph_\omega, \\ &\geqq n+2 && \text{if card } \lambda \geqq \aleph_n, \quad 0 \leqq n < \omega, \\ &\geqq 2 && \text{if card } \lambda \text{ is a nonzero integer.} \end{aligned}$$

THEOREM 6. *Let A be a proper prime ideal in a pli-domain R and let λ be an ordinal such that $A^{[\omega^\lambda]} \neq (0)$. If* card $\lambda =$ card $R = \aleph_n$ *where n is a nonzero integer then*

$$\text{r. gl. dim } R = n + 1; \qquad \text{l. gl. dim } R = 1.$$

If card $\lambda \geqq \aleph_\omega$, *then*

$$\text{r. gl. dim } R = \infty; \qquad \text{l. gl. dim } R = 1.$$

THEOREM 7 (CF. [1]). *Let $1 \leqq m \leqq n \leqq \infty$. Then there exists a left Noetherian domain D such that*

$$\text{r. gl. dim } D = n; \qquad \text{l. gl. dim } D = m.$$

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