RELATIVE GROTHENDIECK RINGS[1]

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1. **Introduction.** Let H be a subgroup of the finite group G, and let Ω be a field of characteristic p, where we assume $p \neq 0$ to avoid trivial cases. Form the free abelian group $\mathcal{A}$ on the symbols $[M]$, where M ranges over the representatives of a full set of isomorphism classes of finitely generated left ΩG-modules (hereafter called "G-modules" for brevity). Let $\mathcal{B}$ be the subgroup of $\mathcal{A}$ generated by all expressions $[M]-[L]-[N]$, where

$$0 \to L \to M \to N \to 0$$

ranges over all H-split exact sequences of G-modules. The *relative Grothendieck ring* $a(G, H)$ is defined as $\mathcal{A}/\mathcal{B}$, acquiring a ring structure by letting $[M][M'] = [M \otimes_{\Omega} M']$ where G acts diagonally on the tensor product.

The structure of $a(G, H)$ has been investigated by the authors in two earlier articles [1], [2]. In the extreme case where $H = 1$, the ring $a(G, 1)$ is just the ring of generalized Brauer characters of G. On the other hand, $a(G, G)$ is the representation ring of G, gotten by considering G-modules relative to direct sum. In general, $a(G, K) \cong a(G, H)$ if H is a Sylow p-subgroup of K, and so there is no loss of generality in assuming hereafter that H is a p-subgroup of G.

Let $k(G, H)$ be the ideal of $a(G, G)$ spanned by all (G, H)-projective G-modules. The *Cartan map*

$$\kappa: k(G, H) \to a(G, H)$$

is defined by $[M] \to [M]$, and as shown in [2], κ is a monomorphism. Furthermore, the cokernel of κ is a p-torsion abelian group when $H \Delta G$.

We have previously established

THEOREM 1 [1, THEOREMS 3.4 AND 4.4]. *If $H \Delta G$, where H is a cyclic p-group, then $a(G, H)$ has a finite free Z-basis. Furthermore, if G is a semidirect product $H \cdot A$, then there is a Z-isomorphism*

$$a(G, H) \cong a(H, H) \otimes_Z a(G, 1).$$

This isomorphism is in fact a ring isomorphism when G is the direct product $H \times A$.

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THEOREM 2 [2, THEOREM 3.2]. *Let G be a semidirect product $P \cdot K$, where P is a normal p-subgroup of G, and let H be any subgroup of K. Then the restriction map $a(G, H) \to a(K, H)$ is a ring isomorphism.*

THEOREM 3 [2, THEOREM 5.1]. *Let G be a direct product $B \times K$, and let H be any subgroup of K. Assume that Ω is a splitting field for G and all of its subgroups. Then there is a ring isomorphism*

$$a(G, H) \cong a(B, 1) \otimes_Z a(K, H).$$

2. **New results.** Making use of the above theorems, we have recently obtained several interesting generalizations, as follows.

THEOREM 4. *Let H be a cyclic subgroup of a p-group G. Then the restriction map $a(G, H) \to a(H, H)$ is a ring isomorphism. The cokernel of the Cartan map κ is a finite p-group.*

THEOREM 5. *Let H be a normal p-subgroup of an aribtrary finite group G, and assume that there exists a set of coset representatives of H in G each of which centralizes H. Then there is a ring isomorphism*

$$a(G, H) \cong a(H, H) \otimes_Z a(G, 1),$$

and hence $a(G, H)$ is Z-free.

The proof of Theorem 5 is based on the following special case, interesting in its own right.

THEOREM 6. *Let H be a normal subgroup of the p-group G, and suppose there exists a set of centralizing coset representatives of H in G. Then the restriction map $a(G, H) \to a(H, H)$ is a ring isomorphism.*

By using Theorems 3 and 6, and an argument involving Frobenius functors, one establishes Theorem 5.

3. **The restriction map.** In view of the above results, it is of interest to determine the image of the restriction map

$$\text{res}: a(G, H) \to a(H, H)$$

in as general a situation as possible. Our main result in this direction is

THEOREM 7. *Let H be a normal p-subgroup of G, where either G is a p-group, or where more generally the p-free part of $[G: H]$ is squarefree. Suppose that the underlying field Ω is algebraically closed and of characteristic p. Let T be the additive subgroup of $a(H, H)$ spanned by all self-conjugate H-modules. Then* res $a(G, H) = T$.

A vital lemma in the proofs of Theorems 6 and 7 is as follows.

LEMMA. *Let H be a normal p-subgroup of G, and suppose that G/H is either a p-group or is cyclic. Let V be any self-conjugate absolutely indecomposable H-module. Then there exists a G-module M such that*

$$(M)_H \cong V \oplus \sum{}^{\oplus} V_i,$$

where each V_i is an H-module with $\dim V_i < \dim V$.

REFERENCES

1. T. Y. Lam and I. Reiner, *Relative Grothendieck groups*, J. Algebra **11** (1969), 213–242.

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