

UNIVALENT FUNCTIONS WITH UNIVALENT DERIVATIVES

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1. Introduction. Let S denote as usual the family of normalized functions

$$(1.1) \quad f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots,$$

which are holomorphic and univalent in the unit disc D . We consider functions $f \in S$ such that most derivatives $f^{(k)}$ are univalent in D . Our conclusion is that f must be entire.

Let E denote those functions $f \in S$ which have the property that for each $n \geq 1$, $f^{(n)}$ is univalent in D . Let

$$(1.2) \quad \alpha = \sup \{ |a_2| ; f \in E \}.$$

2. Main results. The first theorem shows that if $f \in E$, then f must be an entire function of exponential type. Similar results under somewhat weaker hypotheses are possible. These are contained in the second theorem, the proof of which will appear elsewhere.

THEOREM 1. *If $f \in E$, then for $n \geq 2$,*

$$(2.1) \quad |a_n| \leq (2\alpha)^{n-1}/n!$$

and

$$(2.2) \quad \pi/2 \leq \alpha < 1.7208.$$

Furthermore, f is an entire function of exponential type such that for every r , we have

$$(2.3) \quad M(r, f) \leq \{ \exp(2\alpha r) - 1 \} / 2\alpha.$$

PROOF. Since $f^{(n)}$ is univalent in D , $a_{n+1} \neq 0$. Define F_n in D by

$$F_n(z) = \frac{f^{(n)}(z) - n!a_n}{(n+1)!a_{n+1}}.$$

Then $F_n \in E$, and

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$$F_n(z) = z + \frac{(n + 2)a_{n+2}}{2a_{n+1}} z^2 + \dots$$

It follows from (1.2) that

$$|a_{n+2}| \leq \frac{2\alpha |a_{n+1}|}{n + 2}.$$

An inductive argument then establishes (2.1). The fact that f is entire follows from (1.1) and (2.1). Further, we have

$$\begin{aligned} |f(z)| &= \left| z + \sum_2^\infty a_n z^n \right| \\ &\leq \sum_{n=1}^\infty \frac{(2\alpha)^{n-1}}{n!} |z|^n \\ &= \{ \exp(2\alpha |z|) - 1 \} / 2\alpha. \end{aligned}$$

In particular, if $z \in D$, then

$$(2.4) \quad |f(z)| < \{ \exp(2\alpha) - 1 \} / 2\alpha.$$

It is known [2, p. 223] that if $f \in E$, $\lambda \geq 1$, and $|f(z)| < M$ for $z \in D$, then $|a_3 - \lambda a_2^2| \leq (4\lambda - 5)M^{-2} - 8(\lambda - 1)M^{-1} + 4\lambda - 3$. Letting $\lambda = 1$, this becomes $|a_2^2 - a_3| \leq 1 - M^{-2}$. Using (2.1) and (2.4), we have

$$(2.5) \quad |a_2|^2 \leq 1 - 4\alpha^2 / (e^{2\alpha} - 1)^2 + 2\alpha^2 / 3.$$

The right-hand expression in (2.5) is independent of f . It follows from (1.2) that

$$\alpha^2 \leq 3(1 - 4\alpha^2 / (e^{2\alpha} - 1)^2).$$

This implies that $\alpha < 1.7208$.

The lower bound in (2.2) is the second coefficient of the function

$$\phi(z) = \frac{1}{\pi} (e^{\pi z} - 1).$$

Since $\phi \in E$, (2.2) is established.

REMARKS. (i) It is easily seen that E is a normal compact family in D , and so there must exist in E a function whose second coefficient is α .

(ii) The converse of this theorem is false. For example, let $\Psi(z) = \sin z$. Then $\Psi \in S$ and (2.1)–(2.4) are satisfied. But Ψ is not in E .

(iii) Suppose $f(z) = z + a_2 z^2 + \dots$ is regular and mean 1-valent in D and that all derivatives $f^{(k)}$ are also mean 1-valent in D . Then

$|a_2| \leq 2$ [1, p. 99] and our argument shows that f is an entire function of exponential type.

THEOREM 2. *Let $f(z)$ be holomorphic in $|z| < R$.*

$$(2.6) \quad f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < R \quad (R > 0).$$

(i) *Let $\{n_p\}_{p=1}^{\infty}$ be an increasing sequence of positive integers such that for each p , $f^{(n_p)}$ is univalent in D . Then*

$$\liminf_{p \rightarrow \infty} (n_1 n_2 \cdots n_p)^{1/n_p} \leq R.$$

In particular, if $n_{p+1} - n_p = o(\log n_p)$ then f is an entire function.

(ii) *Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that for each n , $f^{(n)}$ is univalent in $|z| < \rho_n$. Then*

$$\liminf_{n \rightarrow \infty} (n(\rho_0 \rho_1 \cdots \rho_n)^{1/n}) \leq 4eR.$$

In particular, if

$$\lim_{n \rightarrow \infty} (n(\rho_0 \rho_1 \cdots \rho_n)^{1/n}) = \infty,$$

then f is entire.

3. Functions in E . Every function $f \in E$ must be of the form

$$(3.1) \quad f(z) = z e^{\beta z} \prod_1^{\infty} \left(1 + \frac{z}{a_n}\right) \exp\left(-\frac{z}{a_n}\right)$$

where

$$(3.2) \quad |\beta| \leq \alpha, \quad n(r) = O(r), \quad |a_1| > 1 \quad \text{and} \quad \sum_{|a_n| \leq r} \frac{1}{a_n} = O(1);$$

or of the form

$$(3.3) \quad f(z) = z e^{\beta z} \prod_1^{\infty} \left(1 + \frac{z}{a_n}\right)$$

where

$$(3.4) \quad \sum 1/|a_n| < \infty, \quad |a_1| > 1, \quad \left| \beta + \sum_1^{\infty} \frac{1}{a_n} \right| \leq \alpha.$$

Obviously these conditions are not sufficient for f to be in E .

THEOREM 3. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $1 \leq |a_1| < |a_2| < \cdots$, $\arg a_1 = \arg a_2 = \cdots = \gamma$ (say) and $\sum 1/|a_n| < \infty$. Let $\arg \beta = -\gamma$ if $\beta \neq 0$ and let*

$$(3.5) \quad f(z) = e^{\beta z} \prod_1^{\infty} \left(1 + \frac{z}{a_n}\right), \quad F(z) = zf(z), \quad G(z) = (f(z) - 1)/f^{(1)}(0).$$

Denote by $\{-b_n\}_1^{\infty}$ the zeros of $f'(z)$ and by $\{-B_n\}_1^{\infty}$ the zeros of $F'(z)$.

(i) If

$$(3.6) \quad |\beta| + \sum_1^{\infty} 1/(|b_n| - 1) \leq 1,$$

then f, f', f'', \dots map D univalently onto convex domains and $G \in E$.

(ii) If $|b_1| \geq 1$ and

$$(3.7) \quad |\beta| + \sum_2^{\infty} 1/(|b_n| - 1) \leq 1,$$

then f, f', f'', \dots are all close-to-convex on D and $G \in E$.

(iii) If

$$\left(\sum_1^{\infty} 1/|b_n|\right) + |\beta| \leq \log 2,$$

then $G \in E$.

(iv) If

$$|\beta| + \sum_1^{\infty} 1/(|B_n| - 1) \leq 1,$$

then F, F', \dots map D univalently onto convex domains and $F \in E$.

(v) If $|B_1| \geq 1$ and

$$|\beta| + \sum_2^{\infty} 1/(|B_n| - 1) \leq 1,$$

then F, F', \dots are all close-to-convex on D and $F \in E$.

(vi) If

$$|\beta| + \sum_1^{\infty} 1/|a_n| \leq \frac{\log 2}{2},$$

then $F \in E$.

The proof of this theorem will appear elsewhere. Note that the conditions for (i) are satisfied if $\beta = 0$, $|a_1| \geq 2$, $|a_{n+1}| \geq 2^n |a_n|$ ($n = 1, 2, \dots$). The conditions for (ii) are satisfied if $\beta = 0$, $|a_1| \geq 1$, $|a_{n+1}| \geq (1.525)^n |a_n|$ ($n = 1, 2, \dots$). Note also that the function ϕ in Theorem 1 is of the form (3.1). (See also [3], [4].)

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