# MANIFOLDS HOMEOMORPHIC TO SPHERE BUNDLES OVER SPHERES

BY R. DE SAPIO[1]

Communicated by William Browder, July 9, 1968

**1. Statement of results.** Let $E$ be the total space of a $k$-sphere bundle over the $n$-sphere with characteristic class $\alpha \in \pi_{n-1}(SO_{k+1})$. We consider the problem of classifying, under the relation of orientation preserving diffeomorphism, all differential structures on $E$. It is assumed that $E$ is simply connected, of dimension greater than five, and its characteristic class $\alpha$ may be pulled back to lie in $\pi_{n-1}(SO_k)$ (that is, the bundle has a cross-section). In [1] and [2] we gave a complete classification in the special case where $\alpha = 0$. The more general classification Theorems 1 and 2 below include this special case. The proofs of these theorems are sketched in §2 below; detailed proofs will appear elsewhere. J. Munkres [6] has announced a classification up to concordance of differential structures in the case where the bundle has at least two cross-sections. (It is well known that concordance and diffeomorphism are not equivalent, concordance of differential structures being strictly stronger than diffeomorphism.)

THEOREM 1. *Let $E$ be the total space of a $k$-sphere bundle over the $n$-sphere whose characteristic class[2] $\alpha$ may be pulled back to lie in $\pi_{n-1}(SO_k)$. Suppose that $2 \leqq k < n-1$. Then, under the relation of orientation preserving diffeomorphism, the diffeomorphism classes of manifolds homeomorphic to $E$ are in a one-to-one correspondence with the equivalence classes on the set $(\theta_n/\Phi_n^{k+1}) \times \theta_{n+k}$, where $(A_*^n, U^{n+k})$ and $(B_*^n, V^{n+k})$ are equivalent if and only if $A_*^n = \pm B_*^n$ and there exists $\beta \in \pi_k(SO_{n-1})$ such that $U^{n+k} - V^{n+k} = \tau'_{n,k}(A_*^n \otimes \beta) + \sigma_{n-1,k}(\alpha \otimes \beta)$.*

Theorem 1 is also true in the case where $k = n-1$ and $n$ is odd. The classification in the case where $n-1 \leqq k \leqq n+2$ is essentially the same as the above and is given in Theorem 2 below. Now we establish the notation used in Theorem 1.

NOTATION. Manifolds and diffeomorphisms are of class $C^\infty$. The group of homotopy $n$-spheres under the connected sum operation +

[1] The preparation of this paper was supported in part by National Science Foundation Grant # GP 7036.

[2] *Added in proof.* Assume here and in Proposition 2 that $\alpha$ is of order 2 in $\pi_{n-1}(SO_{k+1})$ in the case where $k < n-3$. This assumption is not made elsewhere.

is denoted by $\theta_n$, and $\Phi_n^{k+1}$ is the subgroup of $\theta_n$ consisting of those homotopy $n$-spheres that embed in $(n+k+1)$-space with a trivial normal bundle. The class of a homotopy $n$-sphere $A^n$ in the group $\theta_n/\Phi_n^{k+1}$ is denoted by $A_*^n$. Now let

$$\sigma_{n-1,k}: \pi_{n-1}(SO_k) \otimes \pi_k(SO_{n-1}) \rightarrow \theta_{n+k},$$
$$\tau_{n,k}: \theta_n \otimes \pi_k(SO_{n-1}) \rightarrow \theta_{n+k}$$

be the pairings defined in [**4**, p. 583]. It is known that these pairings correspond to composition in the stable homotopy groups of spheres. Moreover, it was shown in [**2**] that $\tau_{n,k}(\Phi_n^{k+1} \otimes \pi_k(SO_{n-1}))=0$, provided that $k \geqq 2$, and hence the pairing $\tau_{n,k}$ induces a pairing

$$\tau'_{n,k}: (\theta_n/\Phi_n^{k+1}) \otimes \pi_k(SO_{n-1}) \rightarrow \theta_{n+k} \qquad (k \geqq 2).$$

REMARK. If $k \geqq n-3$, then $\Phi_n^{k+1}=\theta_n$ (see [**1**, Lemma 1]) and hence $\tau_{n,k}=\tau'_{n,k}=0$ for $k \geqq n-3$.

In order to state the result in the case where $n-1 \leqq k \leqq n+2$ we define a function

$$\sigma'_{n,k}: \pi_{n-1}(SO_k) \times \pi_k(SO_n) \rightarrow \theta_{n+k} \tag{1}$$

that is linear in the second variable. The definition of $\sigma'_{n,k}$ is similar to the definition of the pairing $\sigma_{n-1,k}$ and is described in §2 below. Now if $\alpha \in \pi_{n-1}(SO_k)$ is the characteristic class of the bundle $E$, then we define a homomorphism

$$\chi_\alpha: \pi_k(SO_n) \rightarrow \theta_{n+k}$$

by writing, for each $\beta \in \pi_k(SO_n)$,

$$\chi_\alpha(\beta) = \sigma'_{n,k}(\alpha,\beta).$$

THEOREM 2. *Suppose that the characteristic class $\alpha$ of the bundle $E$ may be pulled back to lie in $\pi_{n-1}(SO_k)$. Then, if $1 \leqq n-3 \leqq k \leqq n+2$ and $k \geqq 2$, then the diffeomorphism classes of manifolds homeomorphic to $E$ are in a one-to-one correspondence with the group $\theta_{n+k}/$*Image $\chi_\alpha$.

2. **Outline of proofs.** We give $E$ the "standard" differential structure by making it a smooth $k$-sphere bundle over the standard $n$-sphere $S^n$. It is well known that if a $k$-sphere bundle over the $n$-sphere has a cross-section, then the total space of the bundle has the homology of the product $S^n \times S^k$. The proof of Theorem 1 is divided into the following four propositions. We use the notation $E(A^n)$ to denote the differential $(n+k)$-manifold obtained by making $E$ into a smooth $k$-sphere bundle over a homotopy $n$-sphere $A^n$ in the obvious way (if

$n=4$, then take $A^4$ to be homeomorphic to $S^4$). We assume that $E$ has a cross-section, $n>3$, and $n+k>5$. We also assume that $k\geqq 2$, except in Proposition 4 where we allow $k=1$.

PROPOSITION 1. *If $M$ is a differential $(n+k)$-manifold that is homeomorphic to $E$, then there are homotopy spheres $A^n$ and $U^{n+k}$ such that $M$ is diffeomorphic to $E(A^n)+U^{n+k}$, provided that $k\leqq n+2$.*

SKETCH OF PROOF. Since $E$ is of dimension greater than five and simply connected we can apply the Hauptvermutung of [7] to conclude that there is a PL-homeomorphism $h$: $M\to E$, where the combinatorial structures are compatible with the differential structures. We try to smooth $h$ by applying the obstruction theory of Munkres [5]. If $k<n$, then the first obstruction to deforming $h$ into a diffeomorphism is an element $c(h)$ in $H_n(M;\Gamma_k)$, where $\Gamma_k$ is the group of diffeomorphisms of $S^{k-1}$ modulo those that extend to diffeomorphisms of the $k$-disk $D^k$. Since $H_n(M;\Gamma_k)$ is isomorphic to $\Gamma_k$ we can consider $c(h)$ to be an element of $\Gamma_k$. Now we construct a manifold $M(c(h))$ and a PL-homeomorphism $j$ from $E$ to $M(c(h))$ such that the first obstruction to smoothing $j$ is $-c(h)$. It follows that the first obstruction to smoothing the composition $jh$ is zero and hence we can suppose that $jh$ is a diffeomorphism modulo the $k$-skeleton. The next step is to show that there is a diffeomorphism modulo a point $\phi$: $M(c(h))\to E$, (this is true for $k\leqq n+2$) and hence the composition $h'=\phi jh$ is a diffeomorphism modulo the $k$-skeleton. The first obstruction to smoothing $h'$: $M\to E$ is an element $c(h')$ in $H_k(M;\Gamma_n)\approx\Gamma_n$. Now let $A^n$ be the homotopy $n$-sphere that corresponds to $c(h')$ under the isomorphism $\Gamma_n\approx\theta_n$ $(n\neq 3)$. There is a PL-homeomorphism $j'$ from $E$ to $E(A^n)$. Moreover, the first obstruction to smoothing $j'$ is $-c(h')$ and hence we can assume that the composition $j'h'$ is a diffeomorphism up to a point. It follows that there is a homotopy $(n+k)$-sphere $U^{n+k}$ such that $M$ is diffeomorphic to $E(A^n)+U^{n+k}$. The argument in the case where $n\leqq k\leqq n+2$ is essentially the same. Note that if $n=4$, then the homotopy sphere $A^4$ is homeomorphic and hence diffeomorphic to $S^4$ since $\Gamma_4=0$.

The remaining propositions combine to give a classification of manifolds of the form $E(A^n)+U^{n+k}$.

PROPOSITION 2. *$E(A^n)$ and $E(B^n)$ are diffeomorphic if and only if* $A^n\equiv\pm B^n \bmod \Phi_n^{k+1}$.

The proof of Proposition 2 is similar to the proofs of Lemmas 5 and 6 of [1]. R. Schultz informs me that he has also proved Proposition 1 and Proposition 2.

PROPOSITION 3. *If $E(A^n)+U^{n+k}$ is diffeomorphic to $E(B^n)$, where $A^n$, $B^n$, $U^{n+k}$ are homotopy spheres, then $E(A^n)$ and $E(B^n)$ are diffeomorphic.*

The proof of Proposition 3 is similar to the proof of Lemma 3 of [1].

PROPOSITION 4. *Let $A^n$, $U^{n+k}$ be homotopy spheres such that $1 \leqq k < n-1$. Then, $E(A^n)+U^{n+k}$ is diffeomorphic to $E(A^n)$ if and only if there exists an element $\beta \in \pi_k(SO_{n-1})$ such that*

$$U^{n+k} = \tau_{n,k}(A^n \otimes \beta) + \sigma_{n-1,k}(\alpha \otimes \beta).$$

The proof of Proposition 4 is similar to the proof of Theorem 3.1 of [2] except that the proof here is a bit more complicated since there are two pairings involved rather than just the pairing $\tau_{n,k}$.

Now we give the construction of the function $\sigma'_{n,k}$ of (1) in §1. Let $\gamma: S^{n-1} \to SO_k$ and $\beta: S^k \to SO_n$ be differentiable maps that represent elements in $\pi_{n-1}(SO_k)$ and $\pi_k(SO_n)$, respectively. We can assume that $\beta$ maps the southern hemisphere $D^k_-$ of $S^k$ into the identity of $SO_n$. Define diffeomorphisms $\lambda_\gamma$ and $\mu_\beta$ of $S^{n-1} \times S^k$ by writing, for each $(u, v) \in S^{n-1} \times S^k$, $\lambda_\gamma(u, v) = (u, s\gamma(u) \cdot v)$ and $\mu_\beta(u, v) = (\beta(v) \cdot u, v)$; here the dot denotes the action of the rotation group on the sphere and $s$ denotes the natural inclusion of $SO_k$ in $SO_{k+1}$. It is clear that $\lambda_\gamma(S^{n-1} \times D^k_-) = S^{n-1} \times D^k_-$ and hence it follows that the diffeomorphism $\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma$ of $S^{n-1} \times S^k$ is the identity on $S^{n-1} \times D^k_-$. Now if $B^{n+k}$ is an $(n+k)$-disk in the interior of $D^n \times S^k$, then it follows that the diffeomorphism $\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma$ can be extended to a diffeomorphism of $D^n \times S^k$ − Interior $B^{n+k}$. The diffeomorphism induced on the $(n+k-1)$-sphere $\partial B^{n+k}$ determines an element $\sigma'_{n,k}(\gamma, \beta)$ of $\theta_{n+k}$, and it is not hard to show that this element depends only on the homotopy classes of $\gamma$ and $\beta$. In fact, $\sigma'_{n,k}(\gamma, \beta)$ is the obstruction to extending $\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma$ to a diffeomorphism of $D^n \times S^k$. Since obstructions are additive with respect to compositions and

$$\lambda_\gamma^{-1}\mu_{\beta+\beta'}\lambda_\gamma = (\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma)(\lambda_\gamma^{-1}\mu_{\beta'}\lambda_\gamma),$$

the correspondence $(\gamma, \beta) \to \sigma'_{n,k}(\gamma, \beta)$ is linear in $\beta$.

PROPOSITION 5. *Let $A^n$ and $U^{n+k}$ be homotopy spheres such that $1 \leqq n-3 \leqq k < 2n-3$. Then, $E(A^n)+U^{n+k}$ is diffeomorphic to $E(A^n)$ if and only if there exists an element $\beta \in \pi_k(SO_n)$ such that $U^{n+k} = \chi_\alpha(\beta)$.*

Now Theorem 2 follows by applying Propositions 1, 2, and 5, noting that $\Phi_n^{k+1} = \theta_n$ for $k \geqq n-3$.

We conclude with some remarks on the case where $k > n+2$. Proposition 1 is not true in this case. For example let $\Sigma^{16}$ denote the nonzero element of $\theta_{16} \approx Z_2$. It is known that $\Sigma^{16}$ does not embed in $R^{29}$ with a trivial normal bundle [3, Theorem 1.3]. Suppose that the conclusion of Proposition 1 is true for $S^{12} \times \Sigma^{16}$; that is, suppose that $S^{12} \times \Sigma^{16}$ is diffeomorphic to $(A^{12} \times S^{16}) + U^{28}$ for homotopy spheres $A^{12}$ and $U^{28}$. It is well known that $A^{12} \times S^{16}$ is diffeomorphic to $S^{12} \times S^{16}$ and hence it follows that $S^{12} \times \Sigma^{16}$ and $S^{12} \times S^{16}$ are diffeomorphic up to a point. This implies that $\Sigma^{16}$ embeds in $R^{29}$ with a trivial normal bundle, a contradiction. On the other hand if $k > n+2$, then the characteristic class $\alpha$ may be pulled back to lie in $\pi_{n-1}(SO_{k-2})$. Define homomorphisms $\eta_\alpha: \theta_k \to \theta_{n+k-1}$ and $\eta'_\alpha: \theta_{k+1} \to \theta_{n+k}$ by writing

$$\eta_\alpha(\Sigma^k) = \tau_{k,n-1}(\Sigma^k \otimes \alpha) \quad \text{and} \quad \eta'_\alpha(\Sigma^{k+1}) = \tau_{k+1,n-1}(\Sigma^{k+1} \otimes \alpha)$$

for $\Sigma^k \in \theta_k$ and $\Sigma^{k+1} \in \theta_{k+1}$, respectively. It follows from [2] that $\Phi^n_k \subset \text{Kernel}\, \eta_\alpha$. Moreover, we can show that the number of distinct (nondiffeomorphic) differential structures on $E$ is not greater than the order of Kernel $\eta_\alpha/\Phi^n_k$ times the order of $\theta_{n+k}/\text{Image}\, \eta'_\alpha$. We plan to give the explicit computation at a later date. Finally, it follows from Munkres [6] that the concordance classes of differential structures on $E$ are in a one-to-one correspondence with

$$\theta_n \oplus (\text{Kernel}\, \eta_\alpha) \oplus (\theta_{n+k}/\text{Image}\, \eta'_\alpha).$$

## References


1. R. De Sapio, *Differential structures on a product of spheres*, Comment. Math. Helv. (to appear).

2. ———, *Differential structures on a product of spheres*. II, Ann. of Math. (to appear).

3. W. C. Hsiang, J. Levine and R. H. Szczarba, *On the normal bundle of a homotopy sphere embedded in Euclidean space*, Topology **3** (1965), 173–181.

4. R. Lashof, *Problems in differential and algebraic topology*, Seattle Conference, 1963, Ann. of Math. (2) **81** (1965), 565–591.

5. J. Munkres, *Obstructions to the smoothing of piecewise differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–554.

6. ———, *Concordance of differentiable structures—two approaches*, Michigan Math. J. **14** (1967), 183–191.

7. D. Sullivan, *On the Hauptvermutung for manifolds*, Bull. Amer. Math. Soc. **73** (1967), 598–600.


University of California, Los Angeles, California 90024