REGARDING STOPPING RULES FOR BROWNIAN MOTION AND RANDOM WALKS[1]

BY LEROY H. WALKER

At the Fifth Berkeley Symposium on Mathematical Statistics and Probability, A. Dvoretzky [4] presented a paper on "Certain Optimal Stopping Rules." In this paper he proved the existence of an optimal stopping rule for the following situation: let X_1, X_2, X_3, $\cdots$ be a sequence of independent, identically distributed, real-valued random variables with zero means and unit variances defined on the probability space $(\Omega_1, \mathcal{F}_1, P_1)$. Then for (fixed) $\beta > \frac{1}{2}$, there exists a positive, integer-valued random variable n (called a stopping rule) defined on Ω_1, whose value for any sample point ω is a function only of the observable variables $X_1(\omega), X_2(\omega), \cdots, X_{n(\omega)}(\omega)$ and which realizes the maximum value of $E\{n^{-\beta} \sum_{i=1}^{n} X_i\}$ over all such random variables, i.e. stopping rules. In his proof, Dvoretzky was lead to conjecture concerning the asymptotic behavior of the stopping boundary which provided the definition of his optimal rule. Theorem 1 of this announcement provides an explicit description of the asymptotic behavior of this boundary. Theorem 2 gives an optimal stopping rule for an analogous situation for a continuous time process, namely Brownian motion.

The paper of Y. S. Chow and H. Robbins [2], which initiated work in this area of stopping rule problems, and the paper of H. Teicher and J. Wolfowitz [7] are also related to the results announced herein. Many of the ideas used to achieve these results were suggested by techniques exhibited in the paper by Teicher and Wolfowitz. While completing the writeup of the results announced herein, the work of L. A. Shepp [5] came to the attention of the author, in which Shepp obtained the same results as given by Theorem 1 and Theorem 2 for the case when $\beta = 1$.

Concise statements of these two theorems require the definition of some notation. The sequence X_1, X_2, X_3, $\cdots$ is as suggested above,

and C_1, C_2, C_3, $\cdots$ is a sequence of positive real numbers. Define N to be the collection of all stopping rules, i.e. functions $n: \Omega_1 \rightarrow I_+ \cup \{\infty\}$, where I_+ denotes the set of positive integers, with the properties that for $k \in I_+$ the event $\{n \leq k\}$ belongs to $\mathfrak{F}_1(X_1, \cdots, X_k)$, where $\mathfrak{F}_1(X_1, \cdots, X_k)$ denotes the minimum σ-algebra of elements of $\mathfrak{F}_1$ in which X_1, X_2, $\cdots$, X_k are all measurable functions, and $P_1(\{n < \infty\}) = 1$. Dvoretzky's results, when $C_k = k^{-\beta}$ with $\beta > \frac{1}{2}$, were that $\sup_{n \in N} E\{C_n \sum_{i=1}^{n} X_i\}$ is finite and that there exists a sequence b_1, b_2, b_3, $\cdots$ of positive real numbers with the property that the function $n': \Omega_1 \rightarrow I_+ \cup \{\infty\}$ defined by

$$n'(\omega) = \sup \left\{ k: \sum_{i=1}^{j} X_i(\omega) < b_j \text{ for } j = 1, 2, \cdots, k - 1 \right\}$$

belongs to N and satisfies the equation

$$E\left\{ C_{n'} \sum_{i=1}^{n'} X_i \right\} = \sup_{n \in N} E\left\{ C_n \sum_{i=1}^{n} X_i \right\}.$$

Such an n' is said to be an optimal stopping rule. Dvoretzky's question was, "Does the limit of $b_m/(m)^{1/2}$ as m tends to infinity exist?"

THEOREM 1. *The limit of $b_m/(m)^{1/2}$ exists and is equal to y_0 where y_0 is the unique positive real root of the equation*

$$\int_0^\infty t^{2(\beta-1)} e^{(yt-t^2/2)} dt = y \int_0^\infty t^{(2\beta-1)} e^{(yt-t^2/2)} dt.$$

The proof of this theorem is long and technically involved. One key idea in the proof is that the b's are determined by the relation

$$\sup_{n \in N} E\left\{ C_{m+n}\left(y + \sum_{i=1}^{n} X_i \right) - C_n y \right\} \begin{cases} > 0 & \text{when } y < b_m \\ \leq 0 & \text{when } y \geq b_m. \end{cases}$$

This relation was used by Teicher and Wolfowitz in their work on stopping rules in this situation.

As the first step in the proof, a continuous analog to this discrete stopping rule situation is considered; namely, let W be a real-valued Brownian motion with mean zero and variance parameter one defined on the probability space $(\Omega_2, \mathfrak{F}_2, P_2)$. Let C denote a function defined by $C(t) = \alpha^t$ with $0 < \alpha < 1$ or $C(t) = (1+t)^{-\beta}$ with $\beta > \frac{1}{2}$ for $t \geq 0$. Define T as the collection of all functions $\tau: \Omega_2 \rightarrow R_+ \cup \{\infty\}$, where R_+ denotes the nonnegative real line, with the properties that for any $r \in R_+$ the event $\{\tau \leq r\}$ belongs to $\mathfrak{F}_2(W_s : s \leq r)$, where $\mathfrak{F}_2(W_s : s \leq r)$ denotes the minimum σ-algebra of elements of $\mathfrak{F}_2$ in which W_s is a measurable function for all $0 \leq s \leq r$, and $P_2(\{\tau < \infty\}) = 1$.

THEOREM 2. $\sup_{\tau \in T} E\{C(\tau)W_\tau\}$ is finite and there exists a function $f : R_+ \to R_+$ determined by the relation

$$\sup_{\tau \in T} E\{C(r+\tau)(y+W_\tau) - C(r)y\} \begin{cases} > 0 & \text{when } y < f(r) \\ \leqq 0 & \text{when } y \geqq f(r) \end{cases}$$

such that the function $\tau' : \Omega_1 \to R_+ \cup \{\infty\}$ defined by

$$\tau'(\omega) = \sup\{t : W_s < f(s) \quad \text{for} \quad 0 \leqq s < t\}$$

is an optimal stopping rule, i.e. τ' belongs to T and

$$E\{C(\tau')W_{\tau'}\} = \sup_{\tau \in T} E\{C(\tau)W_\tau\}.$$

When $C(t) = \alpha^t$, $f(r) = ((2 \log(1/\alpha))^{1/2})^{-1}$.

When $C(t) = (1+t)^{-\beta}$, $f(r) = y_0(1+r)^{1/2}$ where y_0 is the same value as given above for the limit of $b_m/(m)^{1/2}$.

The proof that τ' is optimal is accomplished by restricting analysis to stopping rules which belong to a sequence of subsets of T. The subsets are selected such that the range spaces of the considered stopping rules are well-ordered, discrete, and become dense in R_+ as the subset index tends to infinity. The former two properties permit proofs of optimality over the set of restricted stopping rules because a given rule can be modified on a subset of Ω_2 conditioned on stopping at a particular time. Since such modifications are required at most a countable number of times, measurability properties of the resulting rules remain intact. The denseness property insures that the optimal stopping rules over the subsets of T converge to τ', thereby establishing the optimality of τ'. This optimality proof depends only on properties following from the relation determining $f(r)$. The explicit value of $f(r)$ is obtained by using D. A. Darling and A. J. F. Siegert's [3] results concerning the first contact time of a continuous Markov process with a constant boundary y which gives a formula for $E\{C(\tau)W_\tau\}$ as a function of y. The value of y which makes this expression maximum then determines $f(r)$ explicitly.

As the next step in the proof of the result pertaining to the limit of $b_m/(m)^{1/2}$, the relation determining the b's is expressed in terms of a sampled Brownian motion using Skorokhod's representation of sums of independent, identically distributed random variables.[2] This representation in the form given by L. Breiman[3] provides the existence of

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[2] See the theorem on p. 180 of [6] (p. 163 in the English translation).
[3] Theorem 13.6, p. 276, of [1].

a sequence $\overline{X}_1$, $\overline{X}_2$, $\overline{X}_3$, $\cdots$ of independent, identically distributed, pairs of real-valued random variables defined on a probability space $(\overline{\Omega}_1, \overline{\mathfrak{F}}_1, \overline{P}_1)$ such that a sequence $t(1)$, $t(2)$, $t(3)$, $\cdots$ of nonnegative, independent, identically distributed random variables (called sample times) defined on $(\Omega_2 \times \overline{\Omega}_1, \mathfrak{F}_2 \times \overline{\mathfrak{F}}_1, P_2 \times \overline{P}_1)$ can be constructed with the property that X_1, X_1+X_2, $X_1+X_2+X_3$, $\cdots$ and $W_{t(1)}$, $W_{t(1)+t(2)}$, $W_{t(1)+t(2)+t(3)}$, $\cdots$ have the same joint distribution functions. Then the determining relation for the b's is

$$\sup_{n \in N_m} E\left\{C\left(\frac{k+n}{m}\right)(y + W_{\nu(n,m)}) - C\left(\frac{k}{m}\right)y\right\}$$

$$\begin{cases} > 0 & \text{when } y < b_m/(m)^{1/2} \\ \leq 0 & \text{when } y \geq b_m/(m)^{1/2} \end{cases}$$

where N_m and $\nu(n, m)$ are defined as follows. For $j \in I_+$, $m\nu(j, m)$ is the sum of the first j sample times given by the Skorokhod representation theorem when the Brownian motion $W^{(m)}$, which is defined by $W_t^{(m)} = (m)^{1/2}W_{t/m}$, is used. For the definition of N_m, the concept of the past of the process up to time $\nu(j, m)$ is needed. Technically this concept can be expressed as: let $\mathfrak{F}_3(W_s: s \leq \nu(j, m))$ denote the collection of all subsets A of $\Omega_2 \times \overline{\Omega}_1$ such that for every $r \in R_+$,

$$A \cap \{\nu(j,m) \leq r\} \in \mathfrak{F}_3(\overline{X}_1, \cdots, \overline{X}_j; W_s: s \leq r)$$

where $\mathfrak{F}_3(\overline{X}_1, \cdots, \overline{X}_j; W_s: s \leq r)$ is the minimum σ-algebra of elements of $\mathfrak{F}_2 \times \overline{\mathfrak{F}}_1$ in which $\overline{X}_1$, $\overline{X}_2$, $\cdots$, $\overline{X}_j$ and W_s for $0 \leq s \leq r$ are all measurable functions. Then N_m is the collection of all functions $n: \Omega_2 \times \overline{\Omega}_1 \to I_+ \cup \{\infty\}$ with the properties that for every $k \in I_+$ the event $\{n \leq k\}$ belongs to $\mathfrak{F}_3(W_s: s \leq \nu(k, m))$ and $P_2 \times \overline{P}_1(\{n < \infty\}) = 1$. Proof of this form of the relation determining the b's involves proving the existence of an optimal stopping rule of the form

$$n'(\omega) = \sup\{k: y + W_{\nu(j,m)} < g_m(j) \text{ for } j = 1, 2, \cdots, k-1\}$$

where the $g_m: I_+ \to R_+$ is defined by

$$\sup_{n \in N_m} E\left\{C\left(\frac{j+n}{m}\right)(z + W_{\nu(n,m)}) - C\left(\frac{j}{m}\right)z\right\}$$

$$\begin{cases} > 0 & \text{when } z < g_m(j) \\ \leq 0 & \text{when } z \geq g_m(j). \end{cases}$$

Optimal in this case means $n' \in N_m$ and

$$E\left\{C\left(\frac{n'}{m}\right)(y + W_{\nu(n',m)})\right\} = \sup_{n \in N_m} E\left\{C\left(\frac{n}{m}\right)(y + W_{\nu(n,m)})\right\}.$$

As the last step in the proof of the result pertaining to the limit of $b_m/(m)^{1/2}$, the limit of $\sup_{n \in N_m} E\{C(n/m)(y + W_{\nu(n,m)}) - y\}$ as m tends to infinity is shown to exist and to equal $\sup_{\tau \in T} E\{C(\tau)(y + W_\tau) - y\}$. This result then provides a proof of $\lim \inf_{m \to \infty} b_m/(m)^{1/2} \geq y_0$ by assuming that $\lim \inf_{m \to \infty} b_m/(m)^{1/2} < y_0$, which leads directly to a contradiction. The result that $\lim \sup_{m \to \infty} b_m/(m)^{1/2} \leq y_0$ is not as immediate. For this result, the procedure used is to restrict analysis to subsets of N_m and T for which the above limit result still holds and the supremum of $E\{C(\tau)(y + W_\tau) - y\}$ over the subset of T is a strictly decreasing function of y. This property, coupled with the properties of the sequence $g_m(1)$, $g_m(2)$, $g_m(3)$, $\cdots$ used to define the optimal rule n' described in the last paragraph, then leads to a proof of $\lim \sup_{m \to \infty} b_m/(m)^{1/2} \leq y_0$.

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