ON THE APPROXIMATION BY C-POLYNOMIALS[1]

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Communicated by R. C. Buck, June 11, 1968

1. Introduction. Throughout this note C denotes the unit circle and D its interior. It is the object of this note to give a simple unified treatment to the problem of approximation of a zero free holomorphic function in D (uniformly on compact sets of D) and the problem of bounded approximation of a zero free bounded holomorphic function in D by C-polynomials; i.e. polynomials whose zeros lie on C.

It is known [1], [4] that such approximations are possible in regions whose boundaries satisfy certain smoothness conditions, but the methods used in [1] and [4] yield different approximating sequences. In particular our Main Theorem implies Theorem 1 of [4] for a disk and the Main Theorem of [1]. The proof of the main result is followed by an application of our method to the problem of C-continuation of polynomials [2], [3].

MAIN THEOREM. *Let $f(z)=1+c_1z+c_2z^2+\cdots$, be a zero free holomorphic function in D. Then there exists a sequence of C-polynomials assuming the value one at $z=0$ which converges to $f(z)$ uniformly on every compact subset of D. If in addition the function $f(z)$ is bounded in D then the sequence converges to $f(z)$ boundedly.*

2. Proof of the main theorem. The following lemma is easily verified by observing some simple properties of the linear transformation $(1-z\alpha)(z-\bar{\alpha})^{-1}$ and applying Rouché's Theorem.

LEMMA. *If $P(z)$ is a polynomial of degree m which does not vanish in D then the zeros of the polynomial $P(z)+z^pP^*(z)$, where $P^*(z)=z^m\overline{P}(z^{-1})$ all lie on C for $p=0, 1, 2, \cdots$. Furthermore $|P^*(z)|\leqq|P(z)|$ for $|z|\leqq 1$.*

Let $s_n(z)=1+c_1z+c_2z^2+\cdots+c_nz^n$, and let r_k $(k=1, 2, \cdots)$ be any sequence of positive numbers strictly increasing to one. There exists a strictly increasing sequence of positive integers n_k such that $s_{n_k}(z)\neq 0$ and

$$|s_{n_k}(z)-f(z)|<1/k \qquad \text{for } k=1, 2, \cdots$$

and for $|z|<r_k$. Define $t_{n_k}(z)=s_{n_k}(r_kz)$ and $P_{n_k}(z)=t_{n_k}(z)+z^{n_k}t^*_{n_k}(z)$. Since $t_{n_k}(z)$ does not vanish in D it follows by the lemma that the

[1] The author wishes to acknowledge partial support from NSF grant GP-5221.

$P_{n_k}(z)$ are C-polynomials. Furthermore if $|f(z)|<M$ in D then $|P_{n_k}(z)|<2(M+1)$ for all $z\in D$ and all positive integers n_k. It remains to show that the sequence $\{P_{n_k}(z)\}$ converges to $f(z)$ uniformly on the compact sets of D.

Let ρ, $0<\rho<1$ and $\epsilon>0$ be fixed and let $M_\rho=\operatorname{Max}_{|z|=\rho}|f(z)|$. Choose a positive integer k_0 such that $k_0^{-1}<\epsilon/2$, $r_{k_0}>\rho$ and such that

$$|f(zr_k)-f(z)| < \epsilon/2$$

for all $k\geqq k_0$ and all $|z|\leqq\rho$.

For all n_k $(k\geqq k_0)$ and $|z|\leqq\rho$ we have

$$\begin{aligned} (1) \qquad |t_{n_k}(z)-f(z)| = |s_{n_k}(r_kz)-f(z)| &\leqq |s_{n_k}(r_kz)-f(r_kz)| \\ &+ |(f(r_kz)-f(z))| < \epsilon. \end{aligned}$$

Furthermore by (1) and the lemma

$$(2) \qquad |P_{n_k}(z)-t_{n_k}(z)| = |z^{n_k}t^*_{n_k}(z)| \leqq \rho^{n_k}|t_{n_k}(z)| \leqq \rho^{u_k}(M_\rho+\epsilon).$$

Thus combining (1) and (2) we deduce that

$$|P_{n_k}(z)-f(z)| < \epsilon+\rho^{n_k}(M_\rho+\epsilon).$$

Finally letting $n_k\to\infty$

$$\limsup|P_{n_k}(z)-f(z)| \leqq \epsilon.$$

Since ϵ is arbitrary the proof is complete.

It is interesting to note that the construction of the approximating sequence does not depend on the boundedness of the approximated function.

3. Some remarks on the K-continuation of a polynomial. A somewhat related problem to the ones discussed above is the problem of continuing a given polynomial $P(z)$ of degree n (i.e., adding a finite number of terms of the form a_kz^k, $k>n$) such that the resulting polynomial will have all its zeros on a given set K. Gavrilov [2], proved that if $P(0)\neq 0$ this is possible if K is a piecewise smooth Jordan curve containing the origin in its interior. Later [3] he gave a simpler proof for the case $K\equiv C$. However, these existence proofs are not constructive and do not provide an estimate of the degree of the K-continuation of $P(z)$. Obviously the lemma provides the following

THEOREM. *If $P(z)$ is a polynomial of degree m not vanishing in D then $P(z)+z^pP^*(z)$ is a C-continuation of $P(z)$ for all $p>m$.*

The last theorem provides a partial answer to the problem of estimating the degree of a C-continuation of a polynomial $P(z)$.

THEOREM. *Let $P(z)=1+a_1z+\cdots+a_nz^n$ have the zeros $\alpha_1, \cdots, \alpha_p$* ($p\leqq n$) in D. *If $m(\alpha, k)$ denotes the minimal degree of all polynomials $Q_{k,\alpha}(z)$ such that all the zeros of $1-\alpha z+z^kQ_{k,\alpha}(z)$ lie outside D, then $P(z)$ has a C-continuation of degree not exceeding*

$$2\left[(p+1)n+\sum_{k=1}^{p} m(\alpha_k^{-1}, n+1)\right]+1.$$

The question of finding $m(\alpha, k)$ remains still open.

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