A HOMOLOGICAL METHOD FOR COMPUTING CERTAIN WHITEHEAD PRODUCTS

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1. Introduction. In its simplest form the method for calculating the Whitehead product (WP) $\pi_{n_1}(X)\otimes\pi_{n_2}(X)\to\pi_{n_1+n_2-1}(X)$ may be described as follows. Suppose X is embedded in an H-space E so that the pair (E, X) has trivial homotopy groups in dimensions $<n_1+n_2$. Then we prove that the WP $[\alpha_1, \alpha_2]$ of $\alpha_1\in\pi_{n_1}(X)\equiv\pi_{n_1}(E)$ and $\alpha_2\in\pi_{n_2}(X)\equiv\pi_{n_2}(E)$ is the image under a homomorphism $H_{n_1+n_2}(E)\to\pi_{n_1+n_2-1}(X)$ of the Pontrjagin product of $h(\alpha_1)$ and $h(\alpha_2)$ in the homology ring $H_*(E)$, where $h\colon \pi_*(E)\to H_*(E)$ denotes the Hurewicz homomorphism. Thus, to determine $[\alpha_1, \alpha_2]$, it is necessary to know (1) the effect of h on α_1 and α_2, (2) the Pontrjagin product of $h(\alpha_1)$ and $h(\alpha_2)$, (3) the homomorphism $H_{n_1+n_2}(E)\to\pi_{n_1+n_2-1}(X)$.

It is, however, only sometimes possible to find an H-space for which the information (1), (2) and (3) is available. As a first example, consider the classifying space BU_t of the unitary group U_t and the WP

$$\pi_{2r+2}(BU_t)\otimes\pi_{2s+2}(BU_t)\to\pi_{2t+1}(BU_t),\ t=r+s+1.$$

Here we embed BU_t in the H-space BU_∞ and note that the required information is known. In this way we obtain a new proof of a theorem of Bott [1]. For a second example suppose $\pi_i(X)=0$ for $i<n$ and $n<i<2n-1$ and $\pi_n(X)=\pi$, where n is odd. Then X can be embedded in $K(\pi, n)$. The Pontrjagin square in $H_{2n}(\pi, n)$ is zero and so $[\alpha, \alpha]=0$ for any $\alpha\in\pi$. This result is due to Meyer and Stein [8] (see also §3).

We actually generalize the preceding method by considering kth order WP's instead of ordinary WP's and by requiring that there exist a pair (E, A) with A operating on E rather than an H-space E. Our main result Theorem 1 then yields for ordinary WP's ($k=2$) both the assertion of the first paragraph and a theorem of Meyer [4]. For $k>2$ it enables us, in §3, to extend Bott's theorem by computing kth order WP's in $\pi_*(BU_t)$, and to examine in some detail the kth order WP

$$\pi_n(X)\otimes\cdots\otimes\pi_n(X)\to\pi_{kn-1}(X)$$

when $\pi_i(X)=0$ for $i<n$ and $n<i<kn-1$.

Details of these results and other applications will appear elsewhere.

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2. **The Main Theorem.** We recall the definition of a kth order WP [5]. Elements $\alpha_r \in \pi_{n_r}(X)$, $r=1, \cdots, k$, determine $f: V=S^{n_1} \vee \cdots \vee S^{n_k} \to X$. Let $T=T(S^{n_1}, \cdots, S^{n_k})$ denote the subspace of $P=S^{n_1}\times \cdots \times S^{n_k}$ of k-tuples with at least one coordinate at the base point. If

$$N = \sum_{r=1}^{k} n_r, \quad \lambda \in H_N(P, T) \approx Z$$

is a generator and $\hat{f}: T \to X$ an extension of f, then $\hat{f}_\# \partial h^{-1}(\lambda)$ is in $\pi_{N-1}(X)$, where h is the Hurewicz homomorphism, ∂ the boundary homomorphism and $\hat{f}_\#$ the homomorphism induced by $\hat{f}$:

$$H_N(P, T) \underset{\approx}{\overset{h}{\leftarrow}} \pi_N(P, T) \overset{\partial}{\rightarrow} \pi_{N-1}(T) \overset{\hat{f}_\#}{\rightarrow} \pi_{N-1}(X).$$

The *kth order Whitehead product* $[\alpha_1, \cdots, \alpha_k]$ is the (possibly empty) subset $\{\hat{f}_\# \partial h^{-1}(\lambda) \mid \text{for every extension } \hat{f} \text{ of } f\}$ of $\pi_{N-1}(X)$. When $k=2$ the subset $[\alpha_1, \alpha_2]$ consists of a single element, the ordinary WP of α_1 and α_2. We say that a subspace A of a space E *operates* on E if there exists a map $\mu: E\times A \to E$ such that $\mu | E$ is homotopic to the identity map and $\mu | A$ is homotopic to the inclusion. Then μ induces the *generalized Pontrjagin product* $H_*(E)\otimes H_*(A) \to H_*(E)$.

Suppose X is 1-connected and

(a) there exists a pair (E, A) such that A operates on E and the inclusion $i: A \to E$ induces an isomorphism $i_\#: \pi_s(A) \to \pi_s(E)$ for $s = n_2, \cdots, n_k$

(b) there exists a map $X \to E$ such that $\pi_s(X) \to \pi_s(E)$ is an isomorphism for $s<N-1$ and an epimorphism for $s=N-1$. By using the mapping cylinder we may assume that the map $X \to E$ is an inclusion. Then the pair (E, X) is $(N-1)$-connected and so $h: \pi_N(E, X) \to H_N(E, X)$ is an isomorphism. Thus a homomorphism $H_N(E) \to \pi_{N-1}(X)$ can be defined as the composition

$$H_N(E) \overset{j}{\rightarrow} H_N(E, X) \underset{\approx}{\overset{h^{-1}}{\longrightarrow}} \pi_N(E, X) \overset{\partial}{\rightarrow} \pi_{N-1}(X)$$

where j is induced by inclusion and ∂ is the boundary homomorphism.

THEOREM 1. *Under the assumptions stated above, the kth order* WP *set*

$$[\alpha_1, \cdots, \alpha_k] \text{ of } \alpha_r \in \pi_{n_r}(X), \quad r = 1, \cdots, k,$$

is nonempty and one of its elements is

$$\partial h^{-1} j(h(\alpha_1) * h(i_\#^{-1}\alpha_2) * \cdots * h(i_\#^{-1}\alpha_k))$$

where "$*$" denotes the generalized Pontrjagin product.

For the next result assume that X is $(p-1)$-connected. Let X_n denote the nth Postnikov section of X and $X_{q,p+q-2}$ the fibre of $X_{p+q-2} \to X_{q-1}$. Since this fibration is principal, there is an action of $X_{q,p+q-2}$ on X_{p+q-2}. Letting

$$A = X_{q,p+q-2}, \quad E = X_{p+q-2} \quad \text{and} \quad X \to X_{p+q-2}$$

be the projection, we derive Meyer's theorem [**4**] on the WP of $\alpha_1 \in \pi_p(X)$ and $\alpha_2 \in \pi_q(X)$:

COROLLARY 2. $[\alpha_1, \alpha_2] = \partial h^{-1} j(h(\alpha_1) * h(i_\#^{-1}\alpha_2))$.

We note that $\partial h^{-1} j$ can be identified with the transgression

$$H_{p+q}(X_{p+q-2}) \to H_{p+q-1}(F_{p+q-1}) = \pi_{p+q-1}(X)$$

of the fibration

$$F_{p+q-1} \to X_{p+q-1} \to X_{p+q-2}.$$

COROLLARY 3. *If there exists a map of X into an H-space E such that $\pi_s(X) \to \pi_s(E)$ is an isomorphism for $s < N-1$ and an epimorphism for $s = N-1$, then $\partial h^{-1} j(h\alpha_1 * \cdots * h\alpha_k) \in [\alpha_1, \cdots, \alpha_k] \subset \pi_{N-1}(X)$, where "$*$" denotes Pontrjagin product in $H_*(E)$.*

3. **Higher order Whitehead products.** Here we use Corollary 3 to calculate some higher order WP's.

THEOREM 4. *If $\alpha_r \in \pi_{2m_r+2}(BU_t) \approx Z$ and $\gamma \in \pi_{2t+1}(BU_t) \approx Z_{t!}$ are suitable generators, $r = 1, \cdots, k$ and $t = m_1 + \cdots + m_k + k - 1$, then*

$$m_1! \cdots m_k! \gamma \in [\alpha_1, \cdots, \alpha_k] \subset \pi_{2t+1}(BU_t).$$

The proof proceeds by embedding BU_t in BU_∞ and applying Corollary 3. The factorials appear because $h(\alpha_r) = m_r! p_r$, where p_r is a generator of primitive elements in $H_{2m_r+2}(BU_\infty)$ [**3**].

REMARK. For $k = 2$, Theorem 4 provides a new proof of Bott's theorem [**1**] on the WP $\pi_{2r+2}(BU_t) \otimes \pi_{2s+2}(BU_t) \to \pi_{2t+1}(BU_t)$ (or, what is the same thing, the Samelson product $\pi_{2r+1}(U_t) \otimes \pi_{2s+1}(U_t) \to \pi_{2t}(U_t)$), $t = r + s + 1$. In addition, we can prove a result similar to Theorem 4 for the symplectic group Sp_t and retrieve Bott's theorem on Samelson products in $\pi_*(Sp_t)$ [1].

For the remainder assume that $\pi_i(X)=0$ for $i<n$ and $n<i<kn-1$ $(n>1)$ and set $\pi_n(X)=\pi$ and $\pi_{kn-1}(X)=G$. Then the kth order WP of elements of π is a unique element of G. Let $l_*: H_{kn}(\pi, n)\to H_{kn}(G, kn)=G$ be induced by the first Postnikov invariant l of X and denote by $\gamma_k: H_n(\pi, n)=\pi\to H_{nk}(\pi, n)$ the kth divided power in the ring $H_*(\pi, n)$ [2].

THEOREM 5. *Let $\alpha\in\pi_n(X)$ and $s_1, \cdots, s_k$ be any integers.*

(a) *If n is odd, then $[s_1\alpha, \cdots, s_k\alpha]=0$.*

(b) *If n is even, then $[s_1\alpha, \cdots, s_k\alpha]=s_1\cdots s_k k!\, l_*(\gamma_k(\alpha))$.*

The proof consists of embedding X in $K(\pi, n)$ and applying Corollary 3. The necessary information on $H_*(\pi, n)$ is known [2].

COROLLARY 6. *In addition, assume that $\pi=G=Z$ and $l=mb^k$, a multiple of the kth cup product of the basic class $b\in H_n(Z, n)$. Then if n is even, $[s_1\alpha, \cdots, s_k\alpha]=mk!\, s_1\cdots s_k\gamma$ for a generator γ of Z.*

REMARKS. (1) Porter's result [7] on the kth order WP in complex projective $(k-1)$-space follows immediately from Corollary 6 by setting $n=2$ and $m=1$.

(2) Theorem 5 and Corollary 6 provide another way to obtain some of Porter's examples for certain phenomena regarding higher order WP's [6].

(3) For $k=2$ Theorem 5 is a special case of a theorem of Stein [8]. We note that one direction of Stein's theorem can be extended to kth order WP's.

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