C^*-ALGEBRAS OF TRANSLATIONS AND MULTIPLIERS[1]

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1. **Introduction.** In this note we announce several results about C^*-algebras generated by multiplication and translation operators on L^2-spaces of compact abelian topological groups. The main result, for which the proof is indicated, is that such algebras contain no nontrivial compact operators. It follows that no irreducible, separable C^*-subalgebras of such an algebra can be Type I [2]. We also point out that there are *-isomorphisms between such C^*-algebras on the circle and related C^*-algebras of weighted shifts.

2. **Main result.** Let G be a compact abelian topological group with normalized Haar measure $d\nu$ and consider the associated complex Banach spaces $L^1(G)$, $L^2(G)$, $L^\infty(G)$ and the corresponding real Banach spaces of real-valued functions $L^1_R(G)$, $L^2_R(G)$, $L^\infty_R(G)$. For a in G, an operator T_a is defined on $L^2(G)$ by

$$(T_af)(x) = f(xa).$$

For $\phi(x)$ in $L^\infty(G)$ we can define an operator M_ϕ on $L^2(G)$ by

$$(M_\phi f)(x) = \phi(x)\cdot f(x).$$

We denote by $\tau(G)$ the C^*-algebra generated by all T_a and M_ϕ.

LEMMA 1. *Suppose that for $M>0$ and ϕ_n in $L^\infty(G)$, $1\leqq n\leqq k$, there are $a_i^{(n)}$ in G and real $c_i^{(n)}\geqq 0$ with $1\leqq i\leqq m(n)$, $\sum_{i=1}^{m(n)} c_i^{(n)}=1$ and*

$$\left|\sum_{i=1}^{m(n)} c_i^{(n)}\phi_n(xa_i^{(n)})\right| < M$$

for almost all x in G. Then there are real $c_j\geqq 0$ and a_j in G such that $\sum_{i=1}^{m} c_i=1$ and

$$\left|\sum_{j=1}^{m} c_j\phi_n(xa_j)\right| < M$$

for all $1\leqq n\leqq k$ and almost all x.

PROOF. Let j range over all multi-indices $j=(i_1, i_2, \cdots, i_k)$ where $1\leqq i_n\leqq m(n)$. Then taking

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$$c_{(i_1,i_2,\cdots,i_k)} = \overset{(1)}{c_{i_1}}\overset{(2)}{c_{i_2}} \cdots \overset{(k)}{c_{i_k}},$$

$$a_{(i_1,i_2,\cdots,i_k)} = \overset{(1)}{a_{i_1}} \cdots \overset{(k)}{a_{i_k}},$$

$$m = m(1)\cdot m(2) \cdots m(k)$$

gives the desired result.

We are indebted to D. J. Newman for suggesting the proof of

LEMMA 2. *For any ϕ in $L^\infty(G)$ and $\epsilon > 0$, there are $a_1, \cdots, a_m$ in G and real $c_i \geqq 0$ with $\sum_{i=1}^m c_i = 1$ and*

$$\left| \sum_{i=1}^m c_i\phi(xa_i) - \int \phi(x)d\nu(x) \right| < \epsilon$$

for almost all x.

PROOF. By applying Lemma 1 to the real and imaginary parts of ϕ, it clearly suffices to assume that ϕ is in $L_R^\infty(G)$. Since $L_R^\infty(G)$ is the dual space of $L_R^1(G)$, the unit ball of $L_R^\infty(G)$ is compact in the usual weak topology. It follows that the closed convex hull of $\{T_a\phi : a \text{ in } G\}$ in $L_R^\infty(G)$ is compact in the weak topology. Denote this set by K. We wish to show that the constant function $\int\phi(x)d\nu(x)$ is in K. But, if not, then by the separating hyperplane theorem [**3**, p. 59], there is an $f(x)$ in $L_R^1(G)$ with

$$\sup_{a\in G} \int \phi(xa)f(x)d\nu(x) < \int \phi(x)d\nu(x) \int f(x)d\nu(x).$$

Now integrating with respect to $d\nu(a)$ and using Fubini's theorem, we find

$$\begin{aligned} \int f(x)d\nu(x) \int \phi(x)d\nu(x) &= \int f(x)d\nu(x) \int \phi(xa)d\nu(a) \\ &= \int d\nu(a) \int \phi(xa)f(x)d\nu(x) \\ &< \int \phi(x)d\nu(x) \int f(x)d\nu(x) \end{aligned}$$

and this contradiction finishes the proof.

THEOREM 1. *Let G be a compact abelian topological group which is not totally disconnected. Then $\tau(G)$ contains no nonzero compact operators.*

PROOF. It is easy to see that $\tau(G)$ is irreducible. Hence, it is enough [1, p. 85] to show that there is some compact operator not in $\tau(G)$. Our candidate is the operator of orthogonal projection onto the constant functions

$$(Af)(x) = (f, 1)1 = \left(\int f(x) d\nu(x) \right) 1.$$

Thus, we suppose that for arbitrary $\epsilon_1 > 0$, there are ϕ_n in $L^\infty(G)$ and b_n in G so that

$$(*) \qquad \left\| \sum_{n=1}^{k} \phi_n(x)(T_{b_n}f)(x) - \int f(x) d\nu(x) \right\| < \epsilon_1 \|f\|$$

for all f in $L^2(G)$. Now using the fact that for a in G, T_a is unitary, and considering

$$T_a \left(\sum_{n=1}^{k} M_{\phi_n} T_{b_n} \right) T_a^* - T_a A T_a^*,$$

together with (*) yields

$$(**) \qquad \left\| \sum_{n=1}^{k} \phi_n(xa)(T_{b_n}f)(x) - \int f(x) d\nu(x) \right\| < \epsilon_1 \|f\|.$$

For $\epsilon > 0$ and $1 \leqq n \leqq k$, Lemmas 1 and 2 combine to show that there are a_i in G and real $c_i \geqq 0$, $1 \leqq i \leqq m$, such that $\sum_{i=1}^{m} c_i = 1$ and

$$\left| \sum_{i=1}^{m} c_i \phi_n(xa_i) - \int \phi_n(x) d\nu(x) \right| < \epsilon$$

for almost all x. Now using the triangle inequality, it follows from (**) that

$$\left\| \sum_{n=1}^{k} \left[\sum_{i=1}^{m} c_i \phi_n(xa_i) \right] (T_{b_n}f)(x) - \int f(x) d\nu(x) \right\| < \epsilon_1 \|f\|.$$

Since $\epsilon > 0$ was arbitrary, it is now clear that for

$$s_n = \int \phi_n(x) d\nu(x),$$

we have

$$(***) \qquad \left\| \sum_{n=1}^{k} s_n (T_{b_n}f)(x) - \int f(x) d\nu(x) \right\| < \epsilon_1 \|f\|.$$

Applying (***) to $f(x)=1$ gives

$$\left|1-\sum_{n=1}^{k} s_n\right| < \epsilon_1.$$

On the other hand, since G is not totally disconnected, G has a character χ of infinite order [4, p. 47]. Applying (***) to $\chi^r(x)$, $r\neq 0$, gives

$$\left|\sum_{n=1}^{k} s_n[\chi(b_n)]^r\right| < \epsilon_1.$$

We now observe that by a result in elementary number theory, for any $\delta>0$ an integer r can be found so that $r\neq 0$ and

$$\left|\,[\chi(b_n)]^r-1\,\right| < \delta$$

for all $1\leqq n\leqq k$. It is now clear that

$$\left|\sum_{n=1}^{k} s_n\right| < \epsilon_1,$$

and for $\epsilon_1\leqq\frac{1}{2}$ we have a contradiction.

COROLLARY. *If $\mathfrak{A}$ is an irreducible, separable C^*-subalgebra of $\tau(G)$ for G as in Theorem* 1, *then $\mathfrak{A}$ cannot be Type* I [2].

PROOF. This is immediate by the main result of [2].

3. **Other results.** For $G=T^1$, the circle, consider for fixed a of infinite order in T^1, the C^*-algebra $\mathfrak{A}$ generated by T_a and $\{M_\phi:\phi$ continuous on $T^1\}$. It is clear that $\mathfrak{A}$ is irreducible and separable so the Corollary to Theorem 1 applies to $\mathfrak{A}$. We now introduce a (nonseparable) Hilbert space H with an orthonormal basis $\{\delta_x\}_{x\in T^1}$ indexed by the points of T^1. Thinking of the δ_x as "delta-functions" on T^1, we are led to define operators on H by

$$\tilde{T}_a(\delta_x) = \delta_{xa^{-1}},$$
$$\tilde{M}_\phi(\delta_x) = \phi(x)\delta_x.$$

Now defining $\Phi(T_a)=\tilde{T}_a$ and $\Phi(M_\phi)=\tilde{M}_\phi$, it is easy to check that Φ extends to a *-homomorphism on sums

$$\sum_{n=-k}^{k} M_{\phi_n}T_a^n.$$

THEOREM 2. *The mapping Φ extends to a *-isomorphism between $\mathfrak{A}$ and the C^*-algebra generated by $\tilde{T}_a$ and $\{\tilde{M}_\phi:\phi$ continuous on $T^1\}$.*

The interest in Theorem 2 is that $\Phi(\mathfrak{A})$ is an algebra generated by weighted two-sided shifts. This suggests the possibility of transferring certain computations on operators in $\mathfrak{A}$ to computations in $\Phi(\mathfrak{A})$.

References

1. J. Dixmier, *Les C*-algebras et leurs representations*, Gauthier-Villars, Paris, 1964.
2. J. Glimm, *Type* I *C*-algebras*, Ann of Math. **73** (1961), 572–612.
3. M. Naimark, *Normed rings*, GITTL, Moscow, 1956; English transl., Noordhoff, Groningen, 1959.
4. W. Rudin, *Fourier analysis on groups*, Interscience, New York, 1962.

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