THE IMPOSSIBILITY OF DESUSPENDING COLLAPSES

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It is known that in order to prove the polyhedral Schoenflies conjecture in all dimensions, it is enough to show that, if (B^4, B^3) is a (4, 3) ball pair, then B^4 collapses (polyhedrally) to B^3. Recently, using the solution to the polyhedral Poincaré conjecture in high dimensions, Husch has shown [3] that if (B^7, B^6) is a (7, 6) ball pair, then B^7 collapses to B^6. It is tempting to try to prove that B^4 collapses to B^3 by invoking the following conjecture.

CONJECTURE A. If M is a polyhedral manifold, L a submanifold of M and $S(M)\searrow S(L)$, then $M\searrow L$. ($S(X)$ denotes the suspension of X and "$\searrow$" denotes a polyhedral collapse.)

If Conjecture A were true we could suspend a (4, 3) ball pair three times to obtain a (7, 6) ball pair, use Husch's result, and then apply Conjecture A three times in order to desuspend the collapse.

In this note we present a counterexample to Conjecture A, and discuss other conjectures related to the problem of desuspending collapses.

EXAMPLE 1. Let M^4 be a polyhedral 4-manifold, as described in [4] or [5], with the following properties. (a) M^4 is contractible, (b) $\pi_1(\partial M)\neq 0$, (c) $M^4\times I\cong B^5$. Consider $S(M^4)$ as $M^4\times I$ together with a cone on $M^4\times\{0\}$ and another cone on $M^4\times\{1\}$. Thus if v_0 and v_1 are the vertices of these cones,

$$S(M^4) = (M^4 \times I) \cup (v_0*(M^4 \times \{0\})) \cup (v_1*(M^4 \times \{1\})).$$

Now let B^3 be a 3-ball in ∂M^4. Since $M^4\times I$ is a 5-ball, with $B^3\times I$ as a face, there is an elementary collapse

$$M^4 \times I\searrow (M^4 \times \{0\}) \cup (M^4 \times \{1\}) \cup [(\partial M^4 - \mathrm{int}B^3) \times I].$$

Thus there is a collapse

$$S(M^4)\searrow(v_0 * (M^4\times\{0\}))\cup(v_1 * (M^4\times\{1\}))\cup((\partial M^4-\mathrm{int}B^3)\times \mathrm{I}).$$

Now, by collapsing conewise $v_i * (M^4\times\{i\})$ to $v_i * ((\partial M^4-\mathrm{int}B^3)\times\{i\})$, for $i=0$ and 1, we have $S(M^4)\searrow S(\partial M^4-\mathrm{int}B^3)$. However, since $\pi_1(M^4)=0$ and $\pi_1(\partial M^4-\mathrm{int}B^3)\neq 0$, $M^4\not\searrow\partial M^4-\mathrm{int}B^3$. This provides a counter-example to Conjecture A.

REMARK 1. By taking two copies of the above manifold, M_1 and

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M_2, 3-balls B_1 and B_2 in their boundaries and identifying $\partial M_1 - \text{int } B_1$ with $\partial M_2 - \text{int} B_2$, one can obtain a similar counter-example in which $M = M_1 \cup M_2$ is a 4-ball, and $L = \partial M_1 - \text{int} B_1$ is properly imbedded in M.

REMARK 2. By adapting Example 1, the following can be proved. There exists a polyhedron X and a point $x \in X$ such that $S(X) \searrow S(x)$ but X is not collapsible. Take X as $X = M^4 \cup (x * (\partial M^4 - \text{int} B^3))$; i.e. X is the 4-manifold mentioned in Example 1 together with a cone on its boundary less the interior of a 3-ball. Now $S(X) \searrow S(x)$ by a similar argument to that used above. Suppose X is collapsible. Then $X \searrow x$ (as a collapsible polyhedron collapses to any given point). Then X is P.L. homeomorphic to a regular neighbourhood of x in X (regular neighbourhoods in polyhedra are defined and extensively discussed in [2]). $x * (\partial M^4 - \text{int} B^3)$ is such a regular neighbourhood, so by the regular neighbourhood uniqueness theorem [2] there is a P.L. homeomorphism

$$h: X, x \to x * (\partial M^4 - \text{int} B^3), x.$$

Hence restricting h to the points of X which do not have neighbourhoods which are open 4-cells, h maps the 3-sphere $B^3 \cup (x * \partial B^3)$ homeomorphically onto $(\partial M^4 - \text{int} B^3) \cup (x * \partial B^3)$ which is homeomorphic to ∂M^4. This is impossible as $\pi_1(\partial M^4) \neq 0$, and hence X, is not collapsible.

We now turn our attention to a problem involving simplicial collapsing. Bing [1] has given an example of a triangulation of a 3-cell which is not collapsible. One would hope to be able to suspend this triangulation to obtain noncollapsible triangulations of the n-cell. This leads to Conjecture B.

*CONJECTURE B. If K is a complex, L is a subcomplex of K, and $S(K) \overset{s}{\searrow} S(L)$, then $K \overset{s}{\searrow} L$. ("$\overset{s}{\searrow}$" denotes simplicial collapsing.)

We do not know the answer to Conjecture B, but it seems likely that it is false (although it is not difficult to prove it true if K is only two-dimensional). The following question is related to Conjecture B.

QUESTION 1. Is there a complex K, with subcomplexes X and Y such that $K \overset{s}{\searrow} X$, $K \overset{s}{\searrow} Y$, $K \overset{s}{\searrow} X \cup Y$, but $K \overset{s}{\not\searrow} X \cap Y$?

An affirmative answer to Question 1 would provide a counter-example to Conjecture B as follows: Suppose that K, X and Y have the properties stated in Question 1. Consider $S(K)$ as $(a \cup b) * K$ where a and b are two points. Now since $K \overset{s}{\searrow} X$, $S(K) \overset{s}{\searrow} (a * X) \cup (b * K)$. Since $K \overset{s}{\searrow} Y$, $(a * X) \cup (b * K) \overset{s}{\searrow} (a * X) \cup K \cup (b * Y)$. This latter complex collapses simplicially to $(a * X) \cup (b * Y)$ because $K \overset{s}{\searrow} (X \cup Y)$. By collapsing conewise towards a and b,

$$(a * X) \cup (b * Y) \overset{s}{\searrow} (a * (X \cap Y)) \cup (b * (X \cap Y)) = S(X \cap Y).$$

Thus

$$S(K) \searrow^s S(X \cap Y) \text{ but } K \not\searrow^s X \cap Y.$$

Using the manifold M^4 employed in Example 1, it is possible to show as follows that Question 1 would have an affirmative answer if polyhedral collapsing replaced simplicial collapsing.

EXAMPLE 2. Let M^4 be the manifold used in Example 1, and let B^3 be a 3-ball in ∂M^4 as before. Let X and Y be sub-polyhedra of $M^4 \times I$ defined by

$$X = (M^4 \times \{0\}) \cup ((\partial M^4 - \operatorname{int} B^3) \times I)$$
$$Y = (M^4 \times \{1\}) \cup ((\partial M^4 - \operatorname{int} B^3) \times I).$$

Using the product structure of $M^4 \times I$,

$$M^4 \times I \searrow X \quad \text{and} \quad M^4 \times I \searrow Y.$$

Because $M^4 \times I$ is a 5-ball, $M^4 \times I \searrow X \cup Y$, but $M^4 \times I \not\searrow X \cap Y$ since $X \cap Y = ((\partial M^4 - \operatorname{int} B^3) \times I)$ is not simply connected.

QUESTION 2. With M^4, X and Y as in Example 2, is there a triangulation of $M^4 \times I$, triangulating X and Y as subcomplexes, so that $M^4 \times I \searrow^s X$, $M^4 \times I \searrow^s Y$, and $M^4 \times I \searrow^s X \cup Y$?

* *Added in proof.* The answer to Question 2 is "Yes." L. C. Glaser has pointed out that this follows at once from Theorem 7 of J. H. C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proc. London Math Soc. **45** (1939), 243–327. Thus Question 1 has an affirmative answer, and so Conjecture B *is* false.

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