THE COHOMOLOGY OF COMPACT ABELIAN GROUPS

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Communicated by A. Borel, March 25, 1968

There are at least two (if not more) cohomology theories for a compact group G. The first is the space cohomology of the topological space underlying G based on Čech cochains, say; the second is an extension of the algebraic cohomology of finite groups based on, say, the Čech cochains with compact support on the classifying space of G. This note is concerned with the second cohomology for compact abelian groups. However, for the sake of completeness we recall the main results of the first theory [2].

THEOREM 1. *If G is a compact abelian group, R a commutative ring with identity, then the Čech cohomology $\check{H}(G, R)$ is a graded commutative Hopf algebra over R and is naturally isomorphic to the Hopf algebra $R\otimes C(G, \mathbf{Z})\otimes\Lambda(G_0)^\wedge$, where ΛX is the integral exterior algebra over the group X in degree 1 with its natural Hopf algebra structure, and where $C(G, \mathbf{Z})$ is the Hopf algebra of all continuous functions from G into the discrete ring $\mathbf{Z}$.*

COROLLARY 2. *If G is a compact connected abelian group and R a commutative ring with identity, then there is a natural isomorphism of commutative Hopf algebras $\check{H}(G, R)\cong R\otimes\Lambda\hat{G}$.*

DEFINITION 3. Suppose that $E^1(G)\rightarrow \cdots E^n(G)\rightarrow \cdots E(G) = E^\infty(G)$ is a sequence of spaces and injective maps such that (i) $E^n(G)$ is compact for $n<\infty$, (ii) $\check{H}^i(E^n(G), \mathbf{Z})=0$ for $0<i<n$; $n\leqq\infty$, and that (iii) G acts freely on all spaces. Then $B^n(G)$ is called a classifying space up to n (resp. just a classifying space for $n=\infty$), and for an arbitrary R-module A the graded R-module proj $\lim_n \check{H}(B^n(G), A)$ is independent of the particular choice of a system of classifying spaces. If $A=R$ then the ring structure on $\check{H}(B^n(G), R)$ gives the limit a graded R-algebra structure in a natural fashion. The limit will be called $h(G, A)$.

Functorial constructions for classifying spaces have been given by Milnor, Dold and Lashof, Rothenberg and Steenrod. There is a natural morphism $\check{H}(B(G), A)\rightarrow h(G, A)$, but it is not entirely clear whether it is an isomorphism for all compact groups.

[1] Fellow of the Alfred P. Sloan Foundation.

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PROPOSITION 4. *Let G be a compact connected abelian group and R any commutative ring with identity. Then there is an isomorphism of graded commutative Hopf algebras $h(G, R) \cong R \otimes P\hat{G}$, where PX is the integral symmetric algebra over the group X in degree 2 with its standard Hopf algebra structure. In particular, $h(G, \mathbf{Z}) = P\hat{G}$.*

Note. Compare Corollary 2, Proposition 4 and [5, Theorem H*]. The more general case of a not necessarily connected, compact abelian group is considerably more complicated than one might expect after Proposition 4 drawing from analogy with Theorem 1. In order to make a first observation, we remark that for any compact abelian group the exterior algebra $\wedge G$ can be defined so that it has the familiar properties of the discrete case. There is a natural isomorphism $\wedge G \to \wedge G/G_0$.

THEOREM 5. *Let G be a compact abelian group and R any commutative group (resp. ring with identity). Then there are natural injections of graded abelian groups (resp. R-algebras)*

$$\tau_{G,R} : R \otimes P\hat{G} \to h(G, R), \tag{1}$$

$$\rho_{G,R} : \mathrm{Hom}\ (\wedge G, R) \to h(G, R). \tag{2}$$

(In (2) and in (3) below, $\mathrm{Hom}(\wedge G, R)$ *denotes the group of all continuous group morphisms into the discrete group R; the gradation is the obvious one.)*

Consequently, there is a natural morphism of graded abelian groups (resp. R-algebras)

$$\omega_{G,R} : R \otimes P\hat{G} \otimes \mathrm{Hom}\ (\wedge G, R) \to h(G, R). \tag{3}$$

The group morphism $\omega^i_{G,R}$ is bijective for $i = 0, 1, 2$ if R is a principal ideal domain with zero characteristic.

Note that after Theorem 5 there is a natural $R \otimes P\hat{G}$-module structure on $h(G, R)$ via (1), if R is a ring.

DEFINITION 6. Let $\phi: A \to B$ be a morphism of R-modules over some commutative ring with identity, and $P_R B$ the symmetric R-algebra over B in degree 2. Let $E_2(\phi)$ denote the differential bigraded algebra $P_R B \otimes_R \wedge_R A$ with the differential d_ϕ of bidegree $(2, -1)$ characterized by $d_\phi(x \otimes 1) = 0$ and $d_\phi(1 \otimes a) = \phi(a) \otimes 1$ for $a \in A$. Let $E_3(\phi)$ denote the bigraded algebra derived from $E_2(\phi)$ by passing to cohomology.

DEFINITION 7. A standard resolution of a finite abelian group G is an exact sequence

$$0 \to F \xrightarrow{f} F \xrightarrow{\pi} G \to 0$$

of abelian groups in which $f = f_1 \oplus \cdots \oplus f_n$ so that dom f_i = codom $f_i \cong \mathbf{Z}$, and $f_i x = z_i x$ with natural numbers z_i satisfying $z_i | z_{i+1}$, $i = 1, \cdots, n-1$.

Note that every finite abelian group admits such a resolution in an essentially unique way.

LEMMA 8. *Let G be a compact abelian Lie group and let R be a commutative ring with identity. Then there is an isomorphism of graded commutative rings*

$$h(G, R) \cong P(G_0)^\wedge \otimes E_3(\mathrm{Hom}\,(f, R)),$$

where $0 \to F \xrightarrow{f} F \xrightarrow{\pi} G/G_0 \to 0$ is a standard resolution of G/G_0 and where the R-module action on the right is the obvious one defined by the fact that E_3 is an R-algebra.

In point of fact, there is a natural isomorphism $E_3(\mathrm{Hom}(f, R)) \cong H(G/G_0, R)$, where H denotes the algebraic cohomology [**3**]. The properties of the spectral algebras $E_r(\mathrm{Hom}(f, R))$, $r = 2, 3$ are studied extensively by the authors in a forthcoming paper [**3**].

Since $h(-, R)$ transforms projective limits into direct limits, Lemma 8 makes $h(G, R)$ amenable to computation, at least in principle. However, explicit results are not easy to obtain.

COROLLARY 9. *If $0 \to G_0 \xrightarrow{i} G \xrightarrow{\pi} G/G_0 \to 0$ is the exact sequence defined by the identity component G_0 of a compact abelian group G, then $h(\pi, R): h(G/G_0, R) \to h(G, R)$ is injective and $h(i, R): h(G, R) \to h(G_0, R)$ is surjective.*

Despite Lemma 9, in general we do not have $h(G, R) \cong h(G_0, R) \otimes h(G/G_0, R)$, not even for dim $G = 1$, $G_0 = \hat{\mathbf{Q}}$, $R = \mathbf{Z}$. However, if R is a field, then the situation is better:

THEOREM 10. *Let G be a compact abelian group and R a field with prime field K. Then $\omega_{G,R}$ is an isomorphism and there is a natural isomorphism of graded commutative Hopf algebras*

$$(4) \qquad R \otimes P(G_0)^\wedge \otimes P\,\mathrm{Tor}\,(\hat{G}, K) \otimes \Lambda\,\mathrm{Tor}\,(\hat{G}, K) \to h\,(G, R).$$

COROLLARY 11. *If G is a compact abelian group, then $h(G, \mathbf{R}) \cong h(G_0, \mathbf{R}) \cong P_{\mathbf{R}}\,\mathrm{Hom}(\mathbf{R}, G)^* \cong P_{\mathbf{R}}(\mathbf{R} \otimes \hat{G})$, where the asterisk denotes the dual of a vector space.*

COROLLARY 12. *Let G be a compact abelian group. Then there is a natural isomorphism of Hopf algebras*

$$\begin{aligned} h(G, GF(p)) &\cong GF(p) \otimes P(G_0)^\wedge \otimes P\hat{G}/p\hat{G} \otimes \Lambda\,\hat{G}/p\hat{G} \\ &= GF(p) \otimes P(G_0)^\wedge \otimes E_2(e), \end{aligned}$$

where e is the identity map of $\hat{G}/p\hat{G}$. The differential

$$GF(p) \otimes P(G_0)^\wedge \otimes d_e$$

(see Definition 6) corresponds to the Bockstein derivation of $h(G, GF(p))$ *under this isomorphism. Hence there is a natural exact sequence*

$$0 \to p\, h^+(G, \mathbf{Z}) \cap p\text{-socle } h^+(G, \mathbf{Z}) \to p\text{-socle } h^+(G, \mathbf{Z}) \to \operatorname{im} GF(p) \otimes P(G_0)^\wedge \otimes d_e \to 0,$$

where h^+ *designates the ideal generated by the elements of positive degree.*

As far as $h(G, \mathbf{Z})$ in general is concerned, Lemma 8 and the general theory of the spectral algebras E_r enable us to produce in a canonical fashion a minimal subgroup of $h(G, \mathbf{Z})$ which generates $h(G, \mathbf{Z})$ as a $P\hat{G}$-module (and thus almost as a ring):

THEOREM 13. *Let* G *be a compact abelian group and let* $b^i: h^i(G, \mathbf{R}/\mathbf{Z}) \to h^{i+1}(G, \mathbf{Z})$, $i=0, 1, \cdots$, *be the connecting morphism in the long exact sequence arising from the coefficient sequence* $0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to 0$. *(If* G *is totally disconnected, then* b^i *is an isomorphism.) If* ρ *is the morphism given in* (2) *of Theorem* 5, *then the graded subgroup* $M = \mathbf{Z} \oplus \operatorname{im} b\rho_{G,\mathbf{R}/\mathbf{Z}}$ *is a minimal subgroup such that* $h(G, \mathbf{Z}) = (P\hat{G}) \cdot M$ *with the* $P\hat{G}$*-module structure of* $h(G, \mathbf{Z})$ *afforded by* (1) *of Theorem* 5. *The subgroup* $M + h^2(G, \mathbf{Z})$ *generates the ring* $h(G, \mathbf{Z})$. *(Recall* $h^2(G, \mathbf{Z}) \cong \hat{G}$.) *As a graded abelian group,* M *is isomorphic to* $(\Lambda G)^\wedge$ *under* $b\rho_{G,\mathbf{R}/\mathbf{Z}}$ *with a shift in degree. As a* $P\hat{G}$*-module,* $h(G, \mathbf{Z})$ *is torsion free.*

We remind the reader that a totally disconnected compact abelian group G is a direct product of its p-primary components $G(p)$ such that $G(p)$ is a maximal pro-p-subgroup. One of the sample corollaries of the theory is

PROPOSITION 12. *A compact abelian group* G *has a compact classifying space if and only if it is totally disconnected and* $G(p)$ *is a product of a (possibly empty) collection of* p*-adic groups for each prime* p.

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