FREE INVOLUTIONS ON HOMOTOPY $(4k+3)$-SPHERES

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In [1] Browder and Livesay defined an invariant $\sigma(T, \Sigma) \in 8\boldsymbol{Z}$ of a free differentiable involution T of a homotopy $(4k+3)$-sphere Σ, $k>0$. It is the precise obstruction to finding an invariant $(4k+2)$-sphere of the involution. In [5] and [6] Medrano showed how to construct free involutions with arbitrary Browder-Livesay invariant on *some* homotopy $(4k+3)$-spheres and hence that there exist infinitely many distinct involutions. In this note we investigate the homotopy spheres constructed by Medrano.

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Let μ generate bP_{4k+4}, $[\Sigma]$ denote the representative of Σ in θ_{4k+3} and γ be an arbitrary element of θ_{4k+3}.

THEOREM. (i) *If* $[\Sigma]=2\gamma$ *then for each* i *there exists a free differentiable involution* T_i *such that* $\sigma(T_i, \Sigma)=16i$.

(ii) *If* $[\Sigma]=\mu+2\gamma$ *then for each* i *there exists a free differentiable involution* T_i *such that* $\sigma(T_i, \Sigma)=8+16i$.

The following is an immediate consequence.

COROLLARY. *Each element of* bP_{4k+4}, *in particular* S^{4k+3}, *admits infinitely many differentiably distinct free involutions.*

For $k=1$ more complete results were obtained by Montgomery and Yang [7] and Hirzebruch [2].

Let $\tau(M)$ denote the index of the manifold M.

LEMMA. *For each* i *there exists a homotopy sphere* Σ_i, *bounding a* π-*manifold* M_i, *and an involution* T_i *such that* $\sigma(T_i, \Sigma_i)=\tau(M_i)=8i$.

PROOF. Consider first the case $i=1$ and recall Medrano's construction. Let T_0: $S^{4k+3}\rightarrow S^{4k+3}$ be the antipodal map and $W\cong \#_8(S^{2k+1}\times S^{2k+1})$ be an invariant handlebody obtained from S^{4k+2} by 4 standard disjoint equivariant surgeries. Let F be the standard framing of S^{4k+3} and let V, V' denote the closures of the complements of W in S^{4k+3}.

Let $\{\alpha_1, \cdots, \alpha_8, \beta_1, \cdots, \beta_8\}$ be a standard basis for $H_{2k+1}(W)$, chosen so that $\alpha_i \in \operatorname{Ker}\{i_*: H_{2k+1}(W)\rightarrow H_{2k+1}(V)\}$ and

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$\beta_i \in \operatorname{Ker}\{i'_*: H_{2k+1}(W) \to H_{2k+1}(V')\}$ and so that the matrix U of the form $x \cdot T_{0*}y$ consists of 1's on the nonprincipal diagonal and 0's elsewhere. Choose new generators (and use the summation convention)

$$\alpha_i^* = p_{ij}\alpha_j + q_{ij}\beta_j, \qquad i = 1, \cdots, 8.$$

The matrices P and Q are connected with the well-known 8×8 matrix H of signature 8 and are given explicitly in [6]. Perform framed surgery on the α_i^* and let (A, f) be the resulting framed cobordism between $(W, F|W)$ and a framed homotopy $(4k+2)$-sphere K.

We assert that $V \cup_W A$ is a $(4k+3)$-disc.

Clearly it is simply connected; moreover it has vanishing homology. In order to see $H_{2k+1}(V \cup_W A) = 0$ notice that Q is equivalent by row and column operations over the integers to the identity matrix.

Thus K is in fact a standard sphere and we can sew on a $(4k+3)$-disc D so that $V \cup_W A \cup_K D$ is a sphere bounding a $(4k+4)$-disc B (see Figure 1).

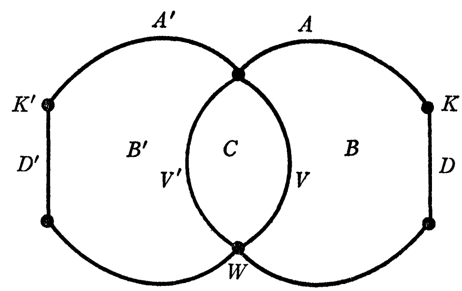

FIGURE 1

Let B' be another copy of B glued to V' by $T_0|V'$, and let C be the standard $(4k+4)$-disc with boundary $S^{4k+3} = V \cup_W V'$ (see Figure 1).

We have obtained a π-manifold $M_1 = B \cup_V C \cup_{V'} B'$ whose boundary is a Medrano homotopy sphere Σ_1 and with the given involution T_1 we have $\sigma(T_1, \Sigma_1) = 8$. It remains to prove that $\tau(M_1) = 8$.

By a Mayer-Vietoris argument it is easy to see that M_1 has (reduced) homology only in dimension $2k+2$, and $H_{2k+2}(M_1)$ is free on 16 generators $\{a_1, \cdots, a_8, b_1, \cdots, b_8\}$ corresponding to the generators of $H_{2k+1}(W)$. We wish to find the intersection matrix of these generators.

Let A_i be a disc in V with boundary α_i and A'_i a disc in $V' \cup_W A'$ with boundary $-\alpha_i$. This specifies the orientation of A_i and A'_i.

Then $A_i \cup A_i'$ represents a_i and similarly $B_i \cup B_i'$ represents b_i, where $B_i' \subset V'$ and $B_i \subset V \cup_W A$. Choose $\partial B_i' = -\partial B_i = \beta_i$. We assert

(1) $$a_i \cdot b_j = \alpha_i \cdot \beta_j = \delta_{ij}.$$

By shifting $B_j \cup B_j'$ along a collar of $B \cup_V C$ we see that $a_i \cdot b_j = A_i \cap B_j' = \alpha_i \cdot \beta_j = \delta_{ij}$. The last equalities assume a preferred orientation of M. This will be used henceforth.

(2) $$b_i \cdot b_j = \bar{q}_{ik} p_{kj} \quad \text{where } (\bar{q}_{rs}) = Q^{-1}.$$

Let A_i^* be a disc bounding α_i^* in A. We have

$$\alpha_i^* = p_{ij}\alpha_j + q_{ij}\beta_j,$$

and hence

$$\beta_j = \bar{q}_{jr}\alpha_r^* - \bar{q}_{jr} p_{rt}\alpha_t.$$

Now we chose $\partial B_i = -\beta_i$, thus we have that B_i is homotopic to the bounded connected sum $\bar{q}_{ir} p_{rt} A_t - \bar{q}_{ir} A_r^*$. Thus b_i can also be represented by $B_i' + \bar{q}_{ir} p_{rt} A_t - \bar{q}_{ir} A_r^*$. Shift the representative $B_j \cup B_j'$ of b_j along a collar of $B \cup_V C$. This shows that

$$b_i \cdot b_j = \bar{q}_{ir} p_{rt} A_t \cap B_j' = \bar{q}_{ir} p_{rt} \delta_{tj} = \bar{q}_{ir} p_{rj}$$

as asserted (see Figure 2).

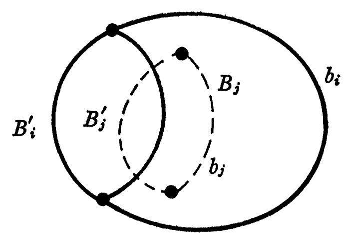

FIGURE 2

(Notice that $Q^{-1}P$ is even and symmetric since $Q^{-1}P = Q^{-1}(PQ^t)(Q^{-1})^t$ and Medrano's construction assumes that PQ^t is even and symmetric.)

A similar analysis shows that $(a_i \cdot a_j) = UQ^{-1}PU$, hence the matrix of intersections is

$$X = \begin{pmatrix} UQ^{-1}PU & I \\ I & Q^{-1}P \end{pmatrix},$$

where

$$Q^{-1}P = \begin{bmatrix} 2 & -1 & -1 & 0 & 1 & -1 & -2 & 1 \\ -1 & 2 & 1 & 1 & -1 & 0 & 3 & -1 \\ -1 & 1 & 2 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & -1 & 2 & -1 \\ 1 & -1 & -1 & 1 & 2 & -1 & -1 & 0 \\ -1 & 0 & 1 & -1 & -1 & 2 & 0 & 0 \\ -2 & 3 & 1 & 2 & -1 & 0 & 6 & -2 \\ 1 & -1 & 0 & -1 & 0 & 0 & -2 & 2 \end{bmatrix}.$$

Now we need to compute the signature of X. Recall that a quadratic form is positive definite if and only if each corner diagonal minor has positive determinant. Since $\det P = \det Q^{-1}P = 0$, X is not positive definite and its signature is not 16. On the other hand there is a 9-dimensional subspace spanned by the coordinate vectors in dimensions 1, 4, 5, 6, 10, 11, 14, 15, 16 on which the form is positive definite, so signature $(X) \neq 0$. Hence we conclude that signature $(X) = \tau(M_1) = 8$ and $[\Sigma_1] = \mu$.

This proves the lemma for $i = 1$.

For $i > 1$ use the following notation. For a given matrix R let $\text{diag}_i(R)$ denote the matrix with i diagonal blocks of R. The matrices U, P, Q, H are as above and in [6].

Let U_i denote the $8i \times 8i$ matrix with $1-s$ on the nonprincipal diagonal and $0-s$ elsewhere. There is an orthonormal matrix S_i with each row and column containing one 1 and all other entries 0, such that

$$U_i = S_i \,\text{diag}_i\,(U) S_i^t.$$

Let

$$H_i = S_i \,\text{diag}_i\,(H) S_i^t,$$

$$P_i = S_i \,\text{diag}_i\,(P) S_i^t,$$

$$Q_i = S_i \,\text{diag}_i\,(Q) S_i^t.$$

Clearly H_i is even, symmetric and unimodular, signature $(H_i) = 8i$ and

(i) $H_i = P_i U_i P_i^t - Q_i U_i Q_i^t$,

(ii) $P_i Q_i^t$ is even and symmetric.

Hence we can perform Medrano's construction and the resulting homotopy sphere Σ_i bounds a π-manifold M_i. The involution T_i has Browder-Livesay invariant $\sigma(T_i, \Sigma_i) = 8i$. In order to compute $\tau(M_i)$

observe that the argument for $i=1$ applies to show that the intersection matrix is

$$X_i = \begin{pmatrix} U_iQ_i^{-1}P_iU_i & I_i \\ I_i & Q_i^{-1}P_i \end{pmatrix}.$$

Now

$$\begin{pmatrix} S_i^t & 0 \\ 0 & S_i^t \end{pmatrix}\begin{pmatrix} U_iQ_i^{-1}P_iU_i & I_i \\ I_i & Q_i^{-1}P_i \end{pmatrix}\begin{pmatrix} S_i & 0 \\ 0 & S_i \end{pmatrix} = \begin{pmatrix} \operatorname{diag}_i(UQ^{-1}PU) & I_i \\ I_i & \operatorname{diag}_i(Q^{-1}P) \end{pmatrix}$$

which becomes $\operatorname{diag}_i(X)$ after reordering rows and columns and hence signature $(X_i)=8i$. This establishes the lemma.

PROOF OF THE THEOREM. For this observe that given (T, Σ) we can equivariantly add two copies of any homotopy sphere Σ' and this yields an involution (T^*, Σ^*) where $[\Sigma^*]=[\Sigma]+2[\Sigma']$ and $\sigma(T, \Sigma) = \sigma(T^*, \Sigma^*)$.

QUESTION. According to computations of the stable stems (see May [4]) there are examples of homotopy $(4k+3)$-spheres not of the form $\mu+2\gamma$ or 2γ. Do such spheres admit *any* free involution?

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