CELL-LIKE MAPPINGS OF ANR'S

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We introduce here the concept of "cell-like" mappings, i.e. mappings with "cell-like" inverse sets (definition below). For maps of ANR's, this concept is the natural generalization of cellular maps of manifolds (see (3) below). Also, a proper mapping of ANR's is cell-like if, and only if, the restriction to any inverse open set is a proper homotopy equivalence. This latter condition is one studied by Sullivan in connection with the Hauptvermutung (see [8]).

DEFINITION. A *space* A *is cell-like* if there is an embedding ϕ of A into some manifold M such that $\phi(A)$ is cellular in M (see [3]). A *mapping* $f: X \to Y$ *is cell-like* if $f^{-1}(y)$ is a cell-like space for each $y \in Y$.

The following technical property is useful in studying cell-like spaces.

PROPERTY (**). A map $\phi: A \to X$ has Property (**) if, for each open set U of X containing $\phi(A)$, there is an open set V of X, with $\phi(A) \subset V \subset U$, such that the inclusion $V \subset U$ is null-homotopic (in U).

The above terminology arose in generalizing McMillan's cellularity criterion [6] to hold for cell-like spaces. S. Armentrout [1] has independently studied this property, calling it "property UV^∞".

To avoid confusion, we will assume that an ANR is a retract of a neighborhood of euclidean space $\boldsymbol{R}^n$.

THEOREM 1. *Let A be a compact, finite-dimensional metric space. Then the following are equivalent*:

(a) *A is cell-like.*

(b) *A has the "fundamental shape" or "Čech homotopy type" of a point, as defined by Borsuk in* [2].

(c) *There exists an embedding of A into some ANR which has Property* (**).

(d) *Any embedding of A into any ANR has Property* (**).

Working independently and from a different point of view, Armentrout has obtained results quite similar to Theorem 1. The proof is not hard. The implications (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) make use only of elementary properties of ANR's; (d)$\Rightarrow$(a) is easy using [6].

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Now we can clarify the concept of cell-like maps of ANR's. Recall that a map is *proper* if preimages of compact sets are compact. A *proper homotopy equivalence* is a homotopy equivalence in which all maps and homotopies can be chosen to be proper.

THEOREM 2. *Let X and Y be ANR's, and let f be a proper mapping of X onto Y. Then the following are equivalent:*

(a) *f is cell-like.*

(b) *For each contractible open subset U of Y, $f^{-1}(U)$ is contractible.*

(c) *For any open subset U of Y, $f|f^{-1}(U): f^{-1}(U) \to U$ is a proper homotopy equivalence.*

The proof of Theorem 2 reduces easily to the following:

LEMMA 2.1. *If $f: X \to Y$ is a proper, cell-like map of ANR's, then f is a proper homotopy equivalence.*

(Notice that a cell-like map is onto, since the empty space is not cell-like.)

A crucial step in the proof of (2.1) is

LEMMA 2.2. *Let the following be given:*

(i) *An ANR X.*

(ii) *A locally compact metric space Y.*

(iii) *A locally finite pair (K, L) of simplicial complexes.*

(iv) *A proper, cell-like map $f: X \to Y$.*

(v) *A proper map $\phi: K \to Y$.*

(vi) *A proper map $\psi: L \to X$ such that $f\psi = \phi | L$.*

(vii) *A continuous function $\epsilon: Y \to (0, \infty)$.*

(viii) *A metric d on Y under which closed, bounded sets are compact.*

Then, there exists a proper map $\Phi: K \to X$ such that $\Phi | L = \psi$ and $d(f\Phi, \phi) \leqq \epsilon\phi$.

T. Price independently and G. Kozlowski obtained versions of (2.2) (see [7] and [9]).

Applications to topological manifolds. It follows immediately from Theorem 2 and [5] that

(1) If $f: M \to N$ is a proper, cell-like map of topological n-manifolds (without boundary), $n \geqq 5$, and if U is an open n-cell in N, then $f^{-1}(U)$ is an open n-cell. One consequence is

(2) If f is as in (1), and if X is a cellular subset of N, then $f^{-1}(X)$ is cellular in M. In particular,

(3) A cell-like map of high-dimensional topological manifolds is cellular. (Compare with [4].)

Piecewise linear manifolds. D. Sullivan has studied condition (c) of Theorem 2 in connection with his proof of the Hauptvermutung. In particular, for closed PL manifolds of dimension $\geqq 5$, he can show that any onto map $f: M \to N$ satisfying condition (c) is homotopic to a PL isomorphism: $M \to N$, provided that $\pi_1(M) = H^3(M; Z_2) = 0$. See [8] for a proof.

Added in proof. Although there seems to be some question about the proof in [5] for the case $n=5$, L. C. Siebenmann has recently given a proof of a more general statement which implies (1) as well as

(4) If $f: R^n \to N$ is a proper, cell-like map of R^n onto a manifold, $n \geqq 5$, then $N \approx R^n$.

References

1. S. Armentrout, *Homotopy properties of decomposition spaces*, (to appear.)

2. K. Borsuk, *Homotopy properties of compact subsets of Hilbert space*, Fund. Math. (to appear).

3. M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. 66 (1960), 74–76.

4. V. Kompaniec, *Homotopy properties of cellular mappings of* PL *manifolds*, Ukrain. Mat. Ž. 18 (1966), 3–10.

5. E. Luft, *On contractible open topological manifolds*, Invent. Math. 4 (1967), 192–201.

6. D. R. McMillan, *A criterion for cellularity in a manifold*, Ann. of Math. 79 (1964), 327–337.

7. T. Price, *On decomposition and homotopy*, Notices Amer. Math. Soc. 14 (1967), Abstract 67T-197.

8. C. P. Rourke, *The Hauptvermutung according to Sullivan*, Part II, Mimeographed notes, Princeton University, Princeton, N. J., 1968.

9. G. Kozlowski, *Factorization of certain maps up to homotopy*, (to appear).

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