GLOBAL SOLUTIONS OF HYPERBOLIC SYSTEMS OF CONSERVATION LAWS IN TWO DEPENDENT VARIABLES

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We are interested in general hyperbolic systems of the form

$$(1)\qquad u_t + f(u, v)_x = 0, \qquad v_t + g(u, v)_x = 0$$

with initial data

$$(2)\qquad (v(0, x), u(0, x)) = (v_0(x), u_0(x)).$$

The vector $U=(v, u)$ is a function of t and x, $t \geqq 0$, $-\infty < x < \infty$, and the functions f and g are C^2 functions of two real variables. We assume that the system (1) is hyperbolic in some open set $\mathfrak{U}$ in the v-u plane, with $f_v g_u > 0$. Let $DF(U)$ and $D^2F(U)$ denote respectively the first and second Fréchet derivatives (see [2]) of the vector function $F=(f, g): \mathfrak{U} \to R^2$; and let $r_j(U)$, $j=1, 2$, be the eigenvectors of $DF(U)$, with orthogonal vectors $l_j(U)$, $j=1, 2$: $l_i(U)r_j(U)=0$ for $i \neq j$.

THEOREM 1. *Let the system* (1) *be hyperbolic in an open set* $\mathfrak{U}$ *in the* v-u *plane. Then* (a) *the system* (1) *is genuinely nonlinear in the jth characteristic field at* $U \in \mathfrak{U}$ *(see Lax* [6]*) if and only if*

$$l_j(U) D^2F(U)[r_j(U),\ r_j(U)] \neq 0;$$

(b) *the system* (1) *satisfies the Glimm-Lax shock interaction condition (condition* (c) *of* [4]*) in* $\mathfrak{U}$ *provided that left eigenvectors* $l_j(U)$ *can be chosen so that*

$$l_j(U) D^2F(U)[r_k(U), r_k(U)] > 0, \qquad j, k = 1, 2,\ j \neq k,\ U \in \mathfrak{U}.$$

The Glimm-Lax shock interaction condition states that the interaction of two shocks of one family produces a shock of the same family and a rarefaction wave of the opposite family. Moreover, for sufficiently weak shocks, we are able to prove an analogous theorem for $n \times n$ systems of conservation laws, $n \geqq 2$, which locally admit Riemann invariants. The proof of it uses some ideas in [3].

We assume that the system (1) is genuinely nonlinear in $\mathfrak{U}$, and we normalize r_j by $D\lambda_j(U)[r_j(U)] > 0$, $j=1, 2$, where $\lambda_j = \lambda_j(U)$ is the eigenvalue asociated with r_j, $\lambda_2 > \lambda_1$. We then normalize l_j by $l_j r_j > 0$,

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$j=1, 2$. Our additional assumption on the system (1) is that $l_j(U)D^2F(U)[r_k(U), r_k(U)]>0, \quad j, k=1, 2, U\in\mathcal{U}$.

We define[2] a shock wave of the ith characteristic field $i=1, 2$, to be a discontinuity $x=x(t)$ satisfying the Rankine-Hugoniot condition and the inequality

$$\lambda_i(U(x+0, t)) < \dot{x}(t) < \lambda_i(U(x-0, t)).$$

THEOREM 2. *For each point $P_0=(v_0, u_0)$ in $\mathcal{U}$, there exist two smooth curves through P_0, $u=s(v; P_0)$ and $u=w(v; P_0)$, called the shock and wave curves respectively, defined in $\mathcal{U}$ globally, which consist of states that can be connected to P_0 by a shock wave of the second characteristic field, and a rarefaction wave of the first characteristic field, respectively.*

For each $P_0=(v_0, u_0)$ in $\mathcal{U}$, we require that

$$\sigma(v, u; v_0, u_0) > \lambda_1(v_0, u_0) \tag{3}$$

for all $(v, u)\in\mathcal{U}$ with $u=s(v; P_0)$, where $\sigma(v, u; v_0, u_0)$ is the corresponding shock speed. This requirement is satisfied for example, if any of the following conditions hold in $\mathcal{U}$:

(a) $\lambda_2\geqq 0\geqq\lambda_1$,

(b) $\partial\lambda_1/\partial u\leqq 0$,

(c) $f_{uv}\geqq 0$ and $f_{uu}\leqq 0$ or $g_{vu}\geqq 0$ and $g_{vv}\leqq 0$.

Fix a point $P_0=(v_0, u_0)$ in the v-u plane and let

$$C(P_0) = \{(v, u)\in\mathcal{U}: v\geqq v_0, s(v; P_0)\leqq u\leqq w(v; P_0)\}.$$

These regions $C(P)$ then satisfy the following order condition.

THEOREM 3. *If $P_1\in C(P_0)$, then $C(P_1)\subseteq C(P_0)$.*

To prove this theorem, we first consider the case where $P_1=(v_1, u_1)$ lies on the shock curve starting at P_0; i.e., P_1 satisfies $u_1=s(v_1; P_0)$. If $u_2=s(v_2; P_1)$ is any point on the shock curve from P_1, then we shall show that (v_2, u_2) is not on the shock curve from P_0; i.e., we shall show that $u_2\neq s(v_2; P_0)$. Suppose that this is not the case and let σ_{01}, σ_{02}, σ_{12} be the corresponding shock speeds. Then

$$\sigma_{01}(P_1-P_0) = F(P_1)-F(P_0),$$
$$\sigma_{12}(P_2-P_1) = F(P_2)-F(P_1),$$
$$\sigma_{02}(P_2-P_0) = F(P_2)-F(P_0).$$

Adding the first two equations and comparing with the third shows that

[2] Note that this definition differs slightly from the definition in [6].

$$\sigma_{01}(P_1-P_0)+\sigma_{12}(P_2-P_1)=\sigma_{02}(P_2-P_0)=\sigma_{02}(P_2-P_1)+\sigma_{02}(P_1-P_0).$$

If the vectors P_1-P_0 and P_2-P_1 were not collinear, we would have $\sigma_{01}=\sigma_{02}=\sigma_{12}$ in contradiction to the shock condition $\sigma_{01}>\lambda_2(P_1)>\sigma_{12}$. Hence we conclude that these vectors are collinear so that

$$(u_1-u_0)/(v_1-v_0)=(u_2-u_1)/(v_2-v_1)=(u_2-u_0)/(v_2-v_0).$$

But this too is impossible since we can easily show that the derivative of $(u-u_0)/(v-v_0)$ along the shock curve $u=s(v;\,P_0)$ is positive; i.e., that the shock curve is convex. (We remark that this part of the theorem is proved without using condition (3), and shows that in $\mathfrak{U}$, the interaction of two shocks of the same family produces a shock of the same family plus a rarefaction wave of the opposite family.) For the general case, we first show that the theorem holds if and only if for each $P_1=(v_1,\ w(v_1;\ P_0))$ with $v_1>v_0$, $u=s(v;\ P_1)$ implies $u\geqq s(v;\ P_0)$; i.e., the theorem holds if and only if for every point P_1 on the wave curve through P_0, the shock curve starting at P_1 does not go below the shock curve starting at P_0. We then show that condition (3) implies (actually is equivalent to) this latter condition. We remark that Theorem 3 holds if instead of assuming condition (3), we have a uniqueness theorem for Riemann problems in $C(P_0)$. Hence the theorem will hold, for example, if instead of condition (3), the conditions for uniqueness of "decay of a discontinuity" as described in [7] are satisfied in $\mathfrak{U}$. Thus (3) is a necessary condition if (1) has a unique solution to the Cauchy problem.

In order to prove a global existence theorem for the problem (1), (2), we assume that the initial data satisfies a certain order condition which we now describe. Suppose that the "curve" $u=u_0(x)$, $v=v_0(x)$, $-\infty<x<\infty$, is bounded and contained in $\mathfrak{U}$. Our order condition states that if we let $(v_i,\ u_i)$, $i=1,\ 2$ be two points on this curve corresponding to the points x_i, $i=1,\ 2$ respectively, where $x_1<x_2$, then the Riemann problem for (1) with initial data

$$\begin{aligned}(v_0(x),\ u_0(x)) &= (v_1,\ u_1), \qquad x<0,\\ &= (v_2,\ u_2), \qquad x>0,\end{aligned}$$

is resolved in $\mathfrak{U}$ by a 2-shock and a 1-rarefaction wave. Under these hypotheses we can prove

THEOREM 4. *The Cauchy problem* (1), (2) *has a global solution contained in* $\mathfrak{U}$.

(Similar theorems can be proved in the case where the data is resolved in $\mathfrak{U}$ by a 1-shock and a 2-rarefaction wave.)

Theorems 2, 3 and 4 are extensions of similar theorems found in [5] and [8] where the cases $f_u = g_v = 0$ and $f_u = g_v = g_{uu} = 0$ respectively, are considered, and $\mathcal{U}$ is the half-space $v > 0$. We prove these theorems by extending and simplifying the methods in [5]. In the proof of theorem 4, we find a solution of (1), (2) as a limit of a sequence of solutions of (1) with step data. We show that these approximating solutions are uniformly bounded and have uniformly bounded variation locally in the sense of Tonelli-Cesari, [1], with respect to two independent (not necessarily orthogonal) directions. It then follows that this sequence is compact in the topology of L_1-convergence on compacta, and therefore a subsequence converges to a solution of the problem (1), (2).

In addition to these theorems, we have proved existence theorems for the problems (1), (2) with the same hypotheses on f, g and the initial data, using the difference scheme introduced by Glimm in [3]. Thus the Glimm scheme can be used to solve certain initial-value problems where the variation of the initial data is arbitrarily large.

The complete proofs of these results will appear elsewhere.

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