A BASIS FOR THE LAWS OF PSL(2,5)

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1. **Introduction.** Although it is known that there is a finite basis for the laws of any finite group (Sheila Oates and M. B. Powell [6]), it is not in general an easy matter to find an explicit basis for the laws of a given finite group. Indeed, the set of laws given below is, as far as we know, the only explicit basis known for the laws of a finite non-abelian simple group.

Before writing down the basis we define the law u_n introduced by L. G. Kovács and M. F. Newman [4]:

$$u_3 = \left[(x_1^{-1}x_2)^{x_{1,2}}, (x_1^{-1}x_3)^{x_{1,3}}, (x_2^{-1}x_3)^{x_{2,3}}\right]$$

and, for $n>3$,

$$u_n = \left[u_{n-1}, (x_1^{-1}x_n)^{x_{1,n}}, \cdots, (x_{n-1}^{-1}x_n)^{x_{n-1,n}}\right].$$

THEOREM A. *The set of laws* (1)–(7) *given below is a basis for the laws of* PSL(2, 5), *the simple group of order* 60.

(1) $x^{30} = 1$

(2) $\{(x^{10}y^{10})^6[x^{10}, y^{10}]^2\}^5 = 1$

(3) $\{((x^6y^{12})^5(x^6y^{18})^5)^3[x^6, y^6]^6\}^6 = 1$

(4) $[x^3, y^3]^{15} = 1$

(5) $\{[x^6y^{10}x^{-6}, y^{-10}][y^{10}, x^6]\}^{10} = 1$

(6) $\{[y^{10}x^6y^{-10}, x^{-6}][y^{10}, x^6]^2\}^6 = 1$

(7) $u_{61} = 1.$

It can be verified by direct calculation that PSL(2, 5) satisfies these laws, so it is sufficient to prove that the variety $\mathfrak{B}$ defined by these laws is contained in the variety $\mathfrak{B}_0$ generated by PSL(2, 5).

2. **Notation.** In notation and terminology we will follow the book of Hanna Neumann [5]; we will also assume familiarity with the results of Chapters 1 and 5 of this book.

3. **Finite soluble groups in $\mathfrak{B}$.**

LEMMA 3.1. *Groups in $\mathfrak{B}$ of prime-power order are elementary abelian.*

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PROOF. Law (1) shows that groups whose orders are powers of 2 are elementary abelian, and that those whose orders are powers of 3 or 5 have exponent 3 or 5 respectively.

Law (2) shows that the commutator of two elements in the same 3-group has order 10, and hence, since it certainly has order 3, it is trivial, and the group is abelian.

Law (3) gives the corresponding result for 5-groups.

LEMMA 3.2. *An element of order p which belongs to the normalizer of a q-subgroup of a group in $\mathfrak{B}$ belongs to its centralizer if p and q take the values* 5 *and* 2, 5 *and* 3, *or* 3 *and* 5.

PROOF. This follows from laws (4), (5) and (6). Consider, for instance, law (5). If y is an element of a 3-subgroup and x an element of order 5 in its normalizer, then the first factor in the law vanishes and we are left with $[y, x]^{10}=1$; but certainly $[y, x]^3=1$ and so $[y, x]=1$, as required.

THEOREM 3.3. *A critical soluble group in $\mathfrak{B}$ belongs to $\mathfrak{A}_2\mathfrak{A}_3$, $\mathfrak{A}_3\mathfrak{A}_2$ or $\mathfrak{A}_5\mathfrak{A}_2$.*

PROOF. Let G be a critical soluble group in $\mathfrak{B}$. Since all its Sylow subgroups are elementary abelian, by Theorem 1.2.6 of P. Hall and G. Higman [2], G has p-length 1 for all primes p. Consider in particular, its upper 5-series. This has the form

$$1 = P_0 \leqq N_0 \leqq P_1 \leqq N_1 = G,$$

where N_0 and N_1/P_1 are groups of order prime to 5 and P_1/N_0 is a 5-group. If S is the Sylow 5-group of P_1 then, by Lemma 3.2, $S \leqq C_G(N_0)$ and so, since $P_1=N_0S$, $P_1=N_0\times S$. But S is characteristic in P_1 and hence normal in G and has trivial intersection with N_0 which is also normal in G. Since G is critical we must have $N_0=1$ or $S=1$.

(i) $N_0=1$. Then P_1 is a normal Sylow 5-subgroup of G. If $|G|$ were divisible by 3, then, by Lemma 3.2, there would be elements outside P_1 which centralized P_1, and this is impossible by Lemma 1.2.3 of [2]. Thus G/P_1 has order a power of 2, and hence, since both G/P_1 and P_1 are elementary abelian, $G\in\mathfrak{A}_5\mathfrak{A}_2$.

(ii) $P_1=1$. Then the order of G is divisible only by powers of 2 and 3, and, since G is critical, it cannot possess both a nontrivial normal 2-subgroup and a nontrivial normal 3-subgroup. If G has no nontrivial normal 2-subgroup then its upper 3-series is

$$1 = P_0 = N_0 < P_1 \leqq N_1 = G$$

where P_1 is a 3-group and G/N_1 a 2-group. Hence $G\in\mathfrak{A}_3\mathfrak{A}_2$. An analogous argument shows that if G has no nontrivial normal 3-subgroup then $G \in \mathfrak{A}_2\mathfrak{A}_3$.

COROLLARY 3.4. *Finite soluble groups in $\mathfrak{B}$ belong to $\mathfrak{B}_0$.*

PROOF. By Corollary 4.2.4 of P. J. Cossey [1], $\mathfrak{A}_5\mathfrak{A}_2$, $\mathfrak{A}_3\mathfrak{A}_2$ and $\mathfrak{A}_2\mathfrak{A}_3$ are generated, respectively, by the dihedral groups of orders 10 and 6, and the alternating group on 4 letters. Since these are all subgroups of PSL(2, 5) the varieties $\mathfrak{A}_5\mathfrak{A}_2$, $\mathfrak{A}_3\mathfrak{A}_2$ and $\mathfrak{A}_2\mathfrak{A}_3$ are subvarieties of $\mathfrak{B}_0$. Thus all critical soluble groups in $\mathfrak{B}$ are in $\mathfrak{B}_0$, and, by induction on the order, we see that any finite soluble group in $\mathfrak{B}$ is in $\mathfrak{B}_0$.

4. Finite nonsoluble groups in $\mathfrak{B}$.

LEMMA 4.1. *The only nonabelian simple group in $\mathfrak{B}$ is* PSL(2, 5).

PROOF. By (4.4) of L. G. Kovács and M. F. Newman [4], the law $u_{61}=1$ implies that, for any group in $\mathfrak{B}$, the index of the centralizer of a chief factor cannot exceed 60, and hence any nonabelian simple group in $\mathfrak{B}$ has order $\leqq 60$. Since PSL(2, 5) is the only simple group with this property the result follows.

THEOREM 4.2. *Every finite group in $\mathfrak{B}$ is of the form*

$$A_1 \times \cdots \times A_r \times S$$

where $A_i \simeq$ PSL(2, 5) $(i=1, \cdots, r)$ *and S is soluble.*

PROOF. Suppose not, and let G be a minimal counterexample, then G is certainly critical and not soluble. We have two cases to consider, according as the monolith σG of G is abelian or nonabelian.

(i) σG nonabelian. Then σG is a direct product of groups isomorphic to PSL(2, 5) and the centralizer of σG in G is 1. Hence G is an automorphism group of σG and so has the form

$$1 \leqq K \leqq G$$

where K is a direct product of groups isomorphic to either PSL(2, 5) or S_5 (the symmetric group on 5 letters) and G/K acts as a transitive permutation group on the direct factors of K. Because of the exponent law, S_5 cannot occur, and so K is a direct product of groups isomorphic to PSL(2, 5) and it follows that $K=\sigma G$. If $G/K\neq 1$, there is an element of prime power order acting nontrivially on σG and so G would not have abelian Sylow subgroups. We deduce that $G=K \simeq$PSL(2, 5).

(ii) σG abelian. By the minimality of G, $G/\sigma(G)$ is a direct product in which at least one factor isomorphic to PSL(2, 5) occurs (since G

is not soluble). Suppose $G/\sigma(G)=H_1/\sigma(G)\times H_2/\sigma(G)$ where $H_1/\sigma(G)\simeq\mathrm{PSL}(2,5)$ and $H_2>\sigma(G)$. Then $H_1<G$ and so is of the form $K_1\times\sigma(G)$ where $K_1\simeq\mathrm{PSL}(2,5)$. Thus K_1 induces automorphisms of H_2 which are trivial on $\sigma(G)$ and on $H_2/\sigma(G)$. But any two such automorphisms commute, and, since the only abelian factor group of K_1 is the trivial group, it follows that K_1 centralizes H_2. Thus $G=K_1\times H_2$ is not critical. Hence we must have $H_1=G$, i.e. $G/\sigma G\simeq\mathrm{PSL}(2,5)$. Now σG is a p-group for $p\in\{2,3,5\}$ and, if it were not central in G, then G would have non-abelian p-subgroups. But, if σG were central in G, then, since $G'\neq 1$, $\sigma G\leqq G'\cap Z(G)$ which is 1 for a group with abelian Sylow subgroups by 3.2 of B. Huppert [3]; so again we have a contradiction.

COROLLARY 4.3. *Finite groups in* $\mathfrak{V}$ *are in* $\mathfrak{V}_0$.

5. **Proof of Theorem A.** Since we have shown that finite groups in $\mathfrak{V}$ are in $\mathfrak{V}_0$, the proof of Theorem A will be complete if we can show that $\mathfrak{V}$ is locally finite, since a variety is determined by its finitely generated groups. Now, the finite groups in $\mathfrak{V}$ on a fixed number of generators have bounded order, and so, if $\mathfrak{V}$ were not locally finite it would contain a nonabelian infinite simple group, contradicting Lemma 4.1.

6. **Remarks.** We have avoided the use of (7) whenever possible, for the reason that we would like to delete it or at least replace it by a set of laws involving only a small number of variables. This however we have been unable to do. We have used it to show that PSL(2, 5) is the only nonabelian finite simple group in $\mathfrak{V}$: this could have been avoided by appealing to the (unpublished) classification of simple groups all of whose Sylow subgroups are abelian. This still leaves the problem of local finiteness of $\mathfrak{V}$. Using arguments similar to those of §3.3 of [6] we can show that there is a set of 5 variable laws which imply local finiteness; we have not been able to find such a set explicitly, however.

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CONTINUITY OF THE VARISOLVENT CHEBYSHEV OPERATOR

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In this note we show that the Chebyshev operator T is continuous at all functions whose best approximations are of maximum degree. Let F be an approximating function unisolvent of variable degree on an interval $[\alpha, \beta]$ and let the maximum degree of F be n. Let P be the parameter space of F. All functions considered will be continuous and for such functions we define the norm

$$\|g\| = \max\{ | g(x) | : \alpha \leqq x \leqq \beta\}.$$

The Chebyshev problem is, for a given continuous function f, to find an element $T(f) = F(A^*, \cdot)$, $A^* \in P$, for which

$$\rho(f) = \inf\{\|f - F(A,\cdot)\| : A \in P\}$$

is attained. Such an element $T(f)$ is called a best Chebyshev approximation to f on $[\alpha, \beta]$. $T(f)$ can fail to exist, but is unique and characterized by alternation if it exists. Definitions and theory are given in [1].

LEMMA 1. *Let $F(A, \cdot)$ be the best approximation to f and F have degree n at A. Let $x_0, \cdots, x_n$ be an ordered set of points on which $f - F(A, \cdot)$ alternates n times. If $\|f-g\| < \delta$ and $\|g - F(B, \cdot)\| \leqq \rho(g) + \delta$ then*

$$(1) \quad (-1)^i[F(B, x_i) - F(A, x_i)] \operatorname{sgn}(f(x_0) - F(A, x_0)) \geqq -3\delta, \quad i = 0, \cdots, n.$$

The lemma can be obtained using arguments similar to those of Rice [2, p. 63].

LEMMA 2. *Let F be of degree n (maximal) at A then for given $\delta > 0$*