AN INVARIANT FOR ALMOST-CLOSED MANIFOLDS

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Communicated by S. Smale, November 21, 1967

1. Let M^n be a compact, oriented, connected, n-dimensional differential manifold with ∂M (boundary M) homeomorphic to the $n-1$ sphere S^{n-1}. Then ∂M represents an element $[\partial M]$ of Γ^{n-1}, the group of differential structures (up to equivalence) on S^{n-1}. We consider the (much studied) problem of expressing $[\partial M]$ in terms of "computable" invariants of M.

Let π_{n-1} be the $n-1$ stem, $J_O\colon \pi_n(\mathrm{BSO})\to\pi_{n-1}$ the classical J-homomorphism, and π'_{n-1} the cokernel of J_O. In [5], a map $P\colon \Gamma^{n-1}\to\pi'_{n-1}$ was defined (see below). We will define an invariant $\Delta(M)$ which is a subset of π'_{n-1} (and often consists of a single element). The main theorem states: $P[\partial M]\in\Delta(M)$.

In a strong sense, the definition of $\Delta(M)$ involves only homotopy theory. Moreover, $\Delta(M)$ seems amenable to computation by standard techniques of algebraic topology. We illustrate this below and, as applications, give explicit examples (1) of a manifold M^n, n odd, with $[\partial M]\neq 0$, and (2) of M^n, n even, with $[\partial M]$ not only $\neq 0$, but in fact with $[\partial M]$ not even contained in $\Gamma^{n-1}(\partial\pi)$, the subgroup in Γ^{n-1} of elements which bound π-manifolds. (Examples of M^n, n even, with $[\partial M]\neq 0$ are of course well known.) Other applications, and detailed proofs, will appear elsewhere.

REMARK 1. By [5], kernel $P=\Gamma^{n-1}(\partial\pi)$. If n is odd, $\Gamma^{n-1}(\partial\pi)=0$, so P is injective, while if $n\equiv 2$ (4), kernel $P\subseteq Z_2$. If $n\equiv 0$ (4), kernel P tends to be large (but see §5).

Let BSO, BSPL, BSTop be the stable classifying spaces for orientable vector bundles, piecewise-linear ($=$PL) bundles, topological bundles. There are maps $J_G\colon \pi_n(\mathrm{BSG})\to\pi_{n-1}$ (G=O, PL, Top) and a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0\to\pi_n(\mathrm{BSO}) & \xrightarrow{f} & \pi_n(\mathrm{BSPL}) & \xrightarrow{g} & \Gamma^{n-1}\to 0 & & \\
\| & & \downarrow J_{\mathrm{PL}} & & & & \\
\pi_n(\mathrm{BSO}) & \xrightarrow{J_O} & \pi_{n-1} & \xrightarrow{q} & \pi'_{n-1}\to 0. & &
\end{array}$$

If $z\in\Gamma^{n-1}$, define $P(z)$ as $q(J_{\mathrm{PL}}(y))$, where $g(y)=z$.

2. **On Thom complexes.** Let β be an oriented (topological) k-disk bundle over a CW-complex X, $T(\beta)$ the Thom complex. If X

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$= Y \cup_d e^n$ ($= Y$ with an n-cell attached by $d: \partial e^n \to Y$), then $T(\beta) = T(\beta | Y) \cup_\phi e^{n+k}$. Also, if $* \in Y$ is the basepoint, we have an inclusion $i: S^k = T(\beta | *) \to T(\beta | Y)$. Assume $k, n \geqq 2$.

PROPOSITION 1. *Let $X = Y \cup e^n$, Y a connected $n-1$ dimensional complex. Let α, β be oriented (topological) k-disk bundles over X with $\alpha | Y$ isomorphic to $\beta | Y$. Let $T(\beta) = T(\beta | Y) \cup_\phi e^{n+k}$, $[\phi] \in \pi_{n+k-1}$ $(T(\beta | Y))$. Suppose $T(\alpha)$ is reducible* [4]. *Then $[\phi] \in$ image $i_*: \pi_{n+k-1}(S^k) \to \pi_{n+k-1}(T(\beta | Y))$.*

REMARK 2. If δ is a k-disk bundle over S^n derived from (α, β) by the difference construction, then in fact $[\phi] = \pm i_* J(\delta)$, where $J = J_{\text{Top}}: \pi_n(\text{BSTop}) \to \pi_{n-1} = \pi_{n+k-1}(S^k)$ (here we assume k is large, although the remark has a nonstable analogue).

REMARK 3. Proposition 1 can be generalized to the case in which $T(\alpha)$ is not necessarily reducible. One then has a statement about the difference of the attaching maps in the two Thom complexes.

3. **Definition of the invariant.** Given M^n as in §1, let M^* be the closed PL manifold $M \cup \text{Cone}(\partial M)$. Let ν_M be the k-dimensional normal bundle of M in Euclidean $n+k$ space (k large). Using the fact that the map $\pi_{n-1}(\text{BSO}) \to \pi_{n-1}(\text{BSPL})$ is injective, one sees that ν_M extends to a vector bundle ν^* on M^*. Let $T(\nu^*) = T(\nu_M) \cup_\phi e^{n+k}$, $[\phi] \in \pi_{n+k-1}(T(\nu_M))$. Apply Proposition 1 with $\alpha = \nu_{\text{PL}}(M^*) = k$-dimensional PL normal bundle of M^*, $\beta = \nu^*$. We conclude that $[\phi] \in$ image $i_*: \pi_{n-1} = \pi_{n+k-1}(S^k) \to \pi_{n+k-1}(T(\nu_M))$. Define $\Delta'(\nu^*) \subseteq \pi_{n-1}$ as $\{y \in \pi_{n-1}: i_*(y) = [\phi]\}$. Let $\Delta(\nu^*) = q(\Delta'(\nu^*)) \subseteq \pi'_{n-1}$. Now $\Delta'(\nu^*)$ depends on the particular vector bundle extension ν^* of ν_M; $\Delta(\nu^*)$, however, does not. We may therefore define:

$$\Delta(M) = \Delta(\nu_M) = \Delta(\nu^*),$$

where ν^* is any vector bundle on M^* extending ν_M.

THEOREM 1. *Let M^n be a compact, oriented, connected, differential n-manifold with ∂M homeomorphic to S^{n-1}. Then $\pm P[\partial M] \in \Delta(M)$.*

PROOF (SKETCH). Let $\nu_{\text{PL}}(M^*)$, ν^* be as above. It can be shown that there is a $y \in \pi_n(\text{BSPL})$ with $g(y) = [\partial M]$ and such that y is a difference bundle for $(\nu_{\text{PL}}(M^*), \nu^*)$. By Remark 2, $T(\nu^*) = T(\nu_M) \cup_\phi e^{n+k}$, where $[\phi] = \pm i_* J(y)$. But $q(J(y)) = P[\partial M]$. Thus $\pm P[\partial M] \in \Delta(M)$.

4. **We give some applications of Theorem 1 (M is always as in §1).**

DEFINITION. A manifold M is of type m with respect to (X, β) if X is a CW-complex with m cells in positive dimensions, β is a vector bundle over X, and there is a map $f\colon M\to X$ with $f^*(\beta)$ stably isomorphic to ν_M.

We consider here manifolds of type one. This class of manifolds is certainly wide enough to be of geometric interest. For example, the following are of type one (with respect to S^i and some $\beta\in\pi_i(\mathrm{BSO})$).

(a) $i-1$ connected M^n, $n=2i$.

(b) $i-1$ connected M^n, $n=2i+1$, $i\not\equiv 1, 2\ (8)$.

(c) The manifolds $M^n(g_1, g_2)$, where $g_1\in\pi_{i-1}(\mathrm{SO}(n-i))$ and $g_2\in\pi_{n-i-1}(\mathrm{SO}(i))$, formed by plumbing an $(n-i)$-disk bundle over S^i (with characteristic map g_1) and an i-disk bundle over S^{n-i} (with characteristic map g_2), provided that the bundle over S^{n-i} is stably trivial.

Suppose M^n is of type one with respect to (S^i, β), and let $j\colon S^{n-1}\to M$ be the inclusion of ∂M into M.

Definition. $\Phi_\beta(M)=\{fj \mid f\colon M\to S^i$ and $f^*(\beta)$ stably isomorphic to $\nu_M\}$. (Thus $\Phi_\beta(M)\subseteq\pi_{n-1}(S^i)$.)

Φ_β appears to be an important invariant for the study of manifolds of type one. We take the view that $\Phi_\beta(M)$ is "known" or computable. This is certainly reasonable for cases (a), (b), (c) above. For example, in cases (a) or (b) one can usually express $\Phi_\beta(M)$ in terms of more standard invariants (Pontryagin classes, behavior of cohomology operations, etc.) and in case (c) we have:

Lemma 1. *Let $M^n=M(g_1, g_2)$ as in* (c). *Then M is of type one with respect to (S^i, g_1), and $J(g_2)\in\Phi_{g_1}(M)$. (Here $J\colon\pi_{n-i-1}(\mathrm{SO}(i))\to\pi_{n-1}(S^i)$.)*

We wish to compute $\Delta(M)$ in terms of $\Phi_\beta(M)$.

Theorem 2. *Let M^n be of type one with respect to (S^i, β), $\beta\in\pi_i(\mathrm{BSO})$. Suppose the composition $xy\in\Phi_\beta(M)$, where $y\in\pi_{n-1}(S^p)$, $x\in\pi_p(S^i)$, $i<p<n-1$; and suppose $x^*(\beta)=0$. Then*

(i) *The Toda bracket $\langle J_0(\beta), S_\beta(x), S(y)\rangle$ is defined.*

(ii) $\pm\Delta(M)\subseteq q\langle J_0(\beta), S_\beta(x), S(y)\rangle$.

Explanation. Here $S\colon\pi_{n-1}(S^p)\to\pi_{n-1-p}$ is the suspension map; $S_\beta\colon\{x\in\pi_p(S^i): x^*(\beta)=0\}\to\pi_{p-i}$ is a certain "twisted" suspension map, which we will not define here.

Remark. Theorem 2 can be generalized; for example, one may replace S^p by an arbitrary complex.

Lemma 2. *For a suitable generator γ of $\pi_4(\mathrm{BSO})$, $S_\gamma\colon\pi_7(S^4)\to\pi_3$ satisfies:*

$S_\gamma(H) = 0$, *H the Hopf map*,

$S_\gamma(t) = S(t)$, *t an element of finite order.*

Recall that $\pi_8(S^4) = Z_2 \oplus Z_2 = \{c\} \oplus \{d\}$, where (in notation of [9]) $c = E\nu' \circ \eta_7$, $d = \nu_4 \circ \eta_7$.

As an illustration of Theorem 2, we have

THEOREM 3. *Let M^9 be of type one with respect to (S^4, γ), γ as in Lemma 2. Recall $\Gamma^8 = Z_2$. Then*

(i) *If* 0 *or* $d \in \Phi_\gamma(M)$, *then* $[\partial M] = 0$.

(ii) *If* c *or* $c + d \in \Phi_\gamma(M)$, *then* $[\partial M] \neq 0$.

PROOF (SKETCH). Suppose that $c \in \Phi_\gamma(M)$. By Theorems 1 and 2, $P[\partial M] \in \Delta(M) \subseteq q\langle J(\gamma), S_\gamma(E\nu'), S(\eta_7)\rangle = q\langle J(\gamma), S(E\nu'), S(\eta_7)\rangle$ (by Lemma 2). Using [9, especially Chapter VI], one calculates that this set is the nonzero element of $\pi_8' = Z_2$. Other cases follow similarly.

EXAMPLES. 1. There is a $z \in \pi_4(\mathrm{SO}(4))$ with $J(z) = c$. Consider the 9-manifold $M(g_1, g_2)$ with $g_1 \in \pi_3(\mathrm{SO}(5))$ stably equal to γ and $g_2 = z$. By Lemma 1, $J(z) = c \in \Phi_\gamma(M)$. By Theorem 3, $[\partial M] \neq 0$.

2. There is a $w \in \pi_6(\mathrm{SO}(4))$ with $J(w) = \alpha_1(4) \circ \alpha_1(7)$ [9, p. 178]. Consider the 11-manifold $M(g_1, g_2)$ with $g_1 \in \pi_3(\mathrm{SO}(7))$ stably equal to γ and $g_2 = w$. As above, one sees that $P[\partial M] \in \Delta(M) = q\langle \alpha_1, \alpha_1, \alpha_1\rangle = q(\beta_1) =$ element of order 3 in $\pi_{10}' = Z_2 \oplus Z_3$. Thus $[\partial M]$ is of order 3 in $\Gamma^{10} = Z_2 \oplus Z_3$.

3. There is a $v \in \pi_{11}(\mathrm{SO}(4))$ with $J(v) = E\nu' \circ \epsilon_7$ [9, p. 66]. Consider the 16-manifold $M(g_1, g_2)$ with $g_1 \in \pi_3(\mathrm{SO}(12))$ stably equal to γ and $g_2 = v$. One sees that $P[\partial M] = q\langle \nu, 2\nu, \epsilon\rangle =$ generator of $\pi_{15}' = Z_2$. Thus $[\partial M] \notin \Gamma^{15}(\partial\pi)$.

REMARK. Let p be an odd prime, and let $n = 2p(p-1) - 1$. It is known that the p-primary component of $\Gamma^{n-1} = Z_p$. Theorem 1 gives good information when applied to the problem of detecting the p-primary component of $[\partial M]$, dim $M = n$. For example, one may show that if the p-primary component of $[\partial M] \neq 0$, then $q_1(M) \neq 0$, q_1 the first (mod p) Wu class. In particular, M can not be $2(p-1)$-connected. It may be conjectured that the generator of the p-primary component of Γ^{n-1} bounds a manifold of the homotopy type of $S^t \vee S^{n-t}$, $t = 2(p-1)$. This is true if $p = 3$ (see Example 2 above).

5. **The case $n = 4k$.** Define $r\colon \Gamma^{n-1} \to Q/Z$ (rationals mod 1) as follows: given $z \in \Gamma^{n-1}$, choose $y \in \pi_n(\mathrm{BSPL})$ with $g(y) = z$. Then put $r(z) = (p_k(y))/b_k$ mod 1, where p_k is the kth rational Pontryagin class and $b_k = p_k(\gamma)$, γ a generator of $\pi_n(\mathrm{BSO})$. (By Bott, $b_k = a_k(2k-1)!$, $a_k = 1$ (k even) or 2 (k odd).)

Define P': kernel $r\to\pi_{n-1}$ as follows: if $r(z)=0$, there is a (unique) y with $g(y)=z$ and $p_k(y)=0$. Put $P'(z)=J_{\mathrm{PL}}(y)$.

LEMMA 3. *The pair* (r, P') *is injective, in the sense that if* $r(z)=0$, *then* $P'(z)$ *is defined, and if* $P'(z)=0$, *then* $z=0$.

PROOF. Assume $r(z)=0$, and let $y\in\pi_n(\mathrm{BSPL})$ satisfy $g(y)=z$, $p_k(y)=0$. Then $J_{\mathrm{PL}}(y)=P'(z)=0$, by assumption. Thus $p_k(y)=J_{\mathrm{PL}}(y)=0$. But this implies $y=0$ (see [2], [3], [8]), so $z=0$.

Now given M^n, define $s(M)$ by

$$s(M) = [p_k(\nu^*) - p_k(\nu_{\mathrm{PL}}(M^*))]/b_k \mod 1,$$

where ν^* is any vector bundle on M^* extending ν_M.

If $s(M)=0$, ν^* may be chosen with $p_k(\nu^*)=p_k(\nu_{\mathrm{PL}}(M^*))$. Then define $\Delta'(M)$ as $\{x\in\pi_{n-1}: i_*(x)=[\phi]\}$, where $T(\nu^*)=T(\nu_M)\cup_\phi e^{n+k}$ (as in §3).

THEOREM 1'. (i) $s(M)=r[\partial M]$. (ii) *If* $s(M)=0$, *then* $\pm P'[\partial M]\in\Delta'(M)$.

REMARK. The invariant r is closely related to Milnor's λ invariant [7]. In fact, $b_k\cdot r(z)=\lambda(z)$, mod 1.

Let d_k be the denominator of $B_k/4k$, B_k the kth Bernoulli number. Let j_k be the order of the image of $J_{\mathrm{O}}: \pi_{4k}(\mathrm{BSO})\to\pi_{4k-1}$. Recall that $j_k=t_kd_k$, $t_k=1$ or 2. In every known case, $t_k=1$ (for example, k odd [1], $k=2$ or 4, or k as in [6]).

In the rest of this section, dim $M=4k$, where $t_k=1$.

THEOREM 4. *Let M be a spin manifold, and suppose* $\Delta(M)=0$. *Then* $[\partial M]=0$ *if and only if* $s(M)=0$ *and* $\hat{A}(M^*)$, *the $\hat{A}$-genus of* M^*, *is integral.*[2]

PROOF. *Necessity* is well known.

Sufficiency. Let $T(\nu^*)=T(\nu_M)\cup_\phi e^{n+k}$, where $p_k(\nu^*)=p_k(\nu_{\mathrm{PL}}(M^*))$. Let $y\in\Delta'(M)$; i.e. let $i_*(y)=[\phi]$. One may show that $\hat{A}(M^*)=e(y)$ mod 1, where e is the invariant of [1]. Also, $\Delta(M)=0$ implies $y\in$image J_{O}. But by [1], $y\in$image J_{O} and $e(y)=0$ imply $y=0$ (if $t_k=1$). Thus $\Delta'(M)=0$ and the theorem follows from Theorem 1' and Lemma 3.

EXAMPLE (KERVAIRE-MILNOR). Suppose ν_M is the trivial bundle. Then $[\partial M]=0$ if and only if $(p_k(M^*))/b_k$ and $\hat{A}(M^*)$ are integral.

PROOF. One sees that $\Delta(M)=0$ and that $s(M)=(p_k(M^*))/b_k$ mod 1. Apply Theorem 4.

[2] We use a definition of $\hat{A}$ which differs from the customary one by a factor of $1/a_k$.

6. Theorem 1 can be improved somewhat. Let $D(\nu_M)$ be the set of all differential structures on the topological manifold M with normal bundle equal to ν_M; by restricting each such structure to ∂M, we obtain a subset $\Gamma(\nu_M)$ of Γ^{n-1}. The argument in the proof of Theorem 1 shows that $\pm P(\Gamma(\nu_M))\subseteq\Delta(\nu_M)$. (Using properties of the map J_{PL} [2], [3], [8], one can sometimes show that this inclusion is an equality.)

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