# SOME REMARKS ON $l-l$ SUMMABILITY

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**Introduction.** If $x$ is a complex number sequence and $A=(a_{nk})$ is an infinite matrix of complex numbers, then $A$ determines a transformation of $x$ into the sequence $Ax$, where $(Ax)_n=\sum_k a_{nk}x_k$. Let $l$ represent the set of complex sequences with finite norm $\|x\|=\sum|x_k|$. If $Ax\in l$ whenever $x\in l$, then $A$ is called an $l-l$ method of summation. Let $l_A$ denote the summability field of $A$, that is, the set of all sequences $x$ such that $Ax\in l$. In [3] an attempt is made to characterize those $l-l$ methods $A$ for which $l_A=l$. In what follows we give counterexamples to Theorems 7 and 9 of [3], announce several positive results related to these theorems, and generalize Theorem 5 of [3] for the class of factorable $l-l$ methods.

1. **Terminology and notation.** Knopp and Lorentz [4] show that the matrix $A$ is an $l-l$ method if and only if $\|A\|<\infty$, where $\|A\| =\sup_k \sum_n|a_{nk}|$. $\|A\|$ is the norm of $A$ as an operator from $l$ to $l$. It is known that $l_A$ inherits a locally convex topology making it an FK space. Moreover, each $f\in l_A'$, the dual space of $l_A$, has the representation

$$f(x) = \sum_n t_n \sum_k a_{nk}x_k + \sum_k \beta_k x_k$$

for some bounded sequences $t$ and $\beta$. An $l-l$ method $A$ is reversible if the equation $y=Ax$ has a unique solution $x$ in $l_A$ for each $y$ in $l$. An $l-l$ method $A$ is called perfect if $l$ is dense in $l_A$ (equivalently, if the set $\Delta=\{e^k: k=1,2,\cdots\}$, where $e^k$ is the sequence having a one in the $k$th coordinate and zeros elsewhere, is fundamental in $l_A$). Let $m_r$ denote the set of all sequences $x$ with finite norm $\|x\|=\sup_m|\sum_{k=1}^m x_k|$. In [3], an $l-l$ method $A$ is called 0-perfect if every sequence $x$ in $m_r\cap l_A$ is a limit point in $l_A$ of the set $l$. Concerning these concepts, Jürimäe makes the following two statements. (In Statement B he omits the assumption of reversibility.)

STATEMENT A [3, THEOREM 7]. If $A$ is an 0-perfect $l-l$ method with $l_A\subseteq m_r$, then $l_A=l$.

STATEMENT B [3, THEOREM 9]. A reversible 0-perfect $l-l$ method $B$ sums a sequence $x\notin l$ if and only if

$$\sup_m \sup_k \left|\sum_{n=1}^m b'_{nk}\right| = \infty, \tag{1}$$

where $B'=(b'_{nk})$ is the two-sided reciprocal for $B$. (Given a matrix $BC$ is a two-sided reciprocal for $B$ if $BC=CB=I$, where $I$ is the identity matrix. Right and left reciprocals are defined analogously.)

2. **Examples.** Let $A=(a_{nk})$ be defined by the set of equations

$$a_{1k}=1 \qquad (k=1,2,\cdots),$$
$$a_{nk}=0 \qquad \text{otherwise}.$$

$A$ is an 0-perfect $l-l$ method (indeed, $A$ is perfect; see [2, p. 361]). Moreover, $l_A=c_r$, where $c_r$ is the set of all sequences $x$ for which the series $\sum_k x_k$ converges. Hence, for 0-perfect $l-l$ methods, the condition $l_A\subseteq m_r$ is clearly not sufficient for $l_A=l$ as Statement A asserts.

Next let $B=(b_{nk})$ be given by the set of equations

$$b_{1k}=1 \qquad (k=1,2,\cdots),$$
$$b_{nn}=b_{n,n+1}=1 \qquad (n=2,3,\cdots),$$
$$b_{nk}=0 \qquad \text{otherwise}.$$

$B$ is one-to-one on $l_A$ because $\{\alpha(-1)^k\}\in l_A\subset c_r$ if and only if $\alpha=0$. Furthermore, given $y\in l$ choose $x_k=\sum_{i=k}^{\infty}(-1)^{i-k}y_i$ for $k=2,3,\cdots$, and $x_1=y_1-\sum_{k=2}^{\infty}\sum_{i=k}^{\infty}(-1)^{i-k}y_i$. Then $Bx=y$ so that $B$ is onto $l$. Since the zero sequence is the only bounded left annihilator of $B$, $B$ is perfect [2, Theorem 2]. Its two-sided reciprocal $B'=(b'_{nk})$ is given by the set of equations

$$b'_{11}=1,$$
$$b'_{1,2k}=-1 \qquad (k=1,2,\cdots),$$
$$b'_{1,2k+1}=0 \qquad (k=1,2,\cdots),$$
$$b'_{n,n+k-1}=(-1)^{k-1} \qquad (k\geqq 1, n\geqq 2),$$
$$b'_{nk}=0 \qquad \text{otherwise}.$$

Clearly, $B'$ does not satisfy Equation (1), yet $l_B$ properly contains $l$ since $\{(-1)^k/k\}\in l_B$; hence, for a reversible 0-perfect $l-l$ method, Equation (1) is not necessary for $l_A=l$ as Statement B asserts.

3. **Comments and results concerning $l_A=l$.** The hypotheses of each of Statements A and B do imply that $l_A\subseteq c_r$. This is not surprising since these hypotheses seem to be more natural for $c_r-c_r$ methods than for $l-l$ methods.

Using a result of Banach [1, p. 49], one can easily show that every reversible $l-l$ method has a unique two-sided reciprocal which is its inverse as an operator from $l_A$ to $l$. This result leads immediately to the following theorem.

THEOREM 1. *Let $A$ be a reversible $l-l$ method. Then $l_A=l$ if and only if its two-sided reciprocal is an $l-l$ method.*

Indeed, since $A'(l)=l_A$, $l_A=l$ if and only if $A'$ is an $l-l$ method.

Using the techniques of FK spaces (see, for example, [5]) one gets the following results concerning left reciprocals.

LEMMA 1. *If the $l-l$ method $A$ has a left $l-l$ reciprocal $B$, then $l$ is closed in $l_A$.*

PROOF. For $x\in l$, $\|x\|=\|(BA)x\|\leqq\|B\|\cdot\|Ax\|$, where $\|B\|<\infty$ since $B$ is $l-l$. Therefore, the $l_A$ topology on $l$ is stronger than the usual topology on $l$.

From this lemma we get the following result.

THEOREM 2. *Let $A$ be a reversible $l-l$ method. Then $l_A=l$ if and only if $A$ has a unique left $l-l$ reciprocal.*

PROOF. If $A$ has a left $l-l$ reciprocal, it is unique if and only if the only bounded sequence which annihilates $A$ from the left is the zero sequence. (For, if one can choose a nonzero bounded sequence $t$ such that $tA=0$, then adding $t$ to any row yields a second left $l-l$ reciprocal. Conversely, if $A$ has two left $l-l$ reciprocals, their difference yields a nonzero bounded sequence which annihilates $A$ from the left.) If $l_A=l$, 0 is the only bounded left annihilator of $A$ [2, Theorem 2, p. 359], and, by Theorem 1, $A$ has a left $l-l$ reciprocal, which is therefore unique. On the other hand, if $A$ has a left $l-l$ reciprocal then, by Lemma 1, $l$ is closed in $l_A$, and if this reciprocal is unique $l$ is dense in $l_A$ [2, Theorem 2, p. 359].

In a paper being prepared, the problem of $l_A=l$ will be further investigated.

**4. Factorable methods.** A matrix $A=(a_{nk})$ is factorable if $a_{nk}=a_nb_k$ for $k\leqq n$ and 0 for $k>n$. Let $E$ be an FK space containing the set of all finite sequences. An $l-l$ method $A$ will be called $E$-perfect if $\Delta$ is fundamental in $E\cap l_A$, where the closure is taken in the $l_A$-topology. For example, $A$ is 0-perfect if and only if $A$ is $m_r$-perfect. The following result is proved in [3, Theorem 5].

THEOREM. *Let $A$ be a reversible $l-l$ method. If for every bounded sequence $t$ the condition $tA=0$ implies the condition*

$$\lim_S \sum_k \left| \sum_{n=1}^{S} (a_{nk} - a_{n,k+1})t_n \right| = 0,$$

*then $A$ is 0-perfect.*

We now show that we may drop the assumption of reversibility for factorable $l-l$ methods.

For any sequence space $E$, let $E^*$ denote the set of sequences $t$ such that $\sum_k t_k x_k$ converges for every $x \in E$.

THEOREM 3. *Let $A$ be a factorable $l-l$ method with $a \in l$. If $b \in E^*$, then $A$ is $E$-perfect.*

PROOF. Let $f \in l'_A$ such that $f=0$ on $l$. Then there exists a bounded sequence $t$ such that for all $x \in l_A$,

$$f(x) = \sum_{n=1}^{\infty} t_n a_n \sum_{k=1}^{n} b_k x_k - \sum_{k=1}^{\infty} b_k x_k \sum_{n=k}^{\infty} t_n a_n.$$

For each $h=1, 2, \cdots$, let

$$F_h(x) = \sum_{n=1}^{\infty} t_n a_n \sum_{k=1}^{n} b_k x_k - \sum_{k=1}^{h} b_k x_k \sum_{n=k}^{\infty} t_n a_n.$$

Then $\lim_h F_h(x) = f(x)$ for each $x \in l_A$.

Now let $\epsilon > 0$ be given and restrict $x$ to $E \cap l_A$. Then

$$| F_h(x) | = \left| \sum_{n=h+1}^{\infty} t_n a_n \sum_{k=h+1}^{n} b_k x_k \right| \leqq \sum_{n=h+1}^{\infty} | t_n a_n | \cdot \left| \sum_{k=h+1}^{n} b_k x_k \right|.$$

Let $M = \sum_{n=1}^{\infty} | t_n a_n |$. If $M=0$, $| F_h(x) | = 0$ for each $h$. If $M \neq 0$, choose $H$ so that for $h \geqq H$, $| \sum_{k=h+1}^{n} b_k x_k | < \epsilon / M$ for all $n \geqq h+1$. Then for $h \geqq H$, $| F_h(x) | < \epsilon$. It follows that $f$ vanishes on $E \cap l_A$, so that $A$ is $E$-perfect.

That the converse of Theorem 3 is false follows by letting $E = s$, the FK space of all sequences, and letting $A$ be given by $a_n = 2^{-n}$ and $b_k = 2^k$. $l_A = l$ since the two-sided reciprocal for $A$ is $l-l$ and yet $b$ is not in $s^*$.

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