MODIFICATION SETS OF DENSITY ZERO[1]

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Communicated by W. Wasow, December 18, 1967

Let R, Z, T denote the real line, the integers, and the unit circle, respectively. A set $E\subset R$ will be called a *modification set in* R if to every $f\in L^1(R)$ there corresponds a singular bounded Borel measure μ on R whose Fourier transform $\hat{\mu}$ coincides with $\hat{f}$ in the complement of E. In other words, the Fourier transform of every absolutely continuous measure can be modified on E alone so that the resulting function is the Fourier transform of a singular measure. Modification sets E in Z are defined similarly: to every $f\in L^1(T)$ there should correspond a bounded singular measure μ on T whose Fourier coefficients satisfy $\hat{\mu}(n)=\hat{f}(n)$ for every integer n which is not in E.

The existence of "small" modification sets in locally compact abelian groups has been established in [1]. However, when applied to Z or R, the theorem of [1] can only yield modification sets of positive (though arbitrarily small) lower density. In the present note this result is improved to yield sets of density zero.

A set $E\subset R$ is said to have density zero if $(2t)^{-1}m(E\cap[-t,t])\to 0$ as $t\to\infty$, where m denotes Lebesgue measure. If $E\subset Z$, the requirement is that the number of elements of E in $[-N, N]$, divided by $2N$, should tend to 0 as $N\to\infty$.

THEOREM 1. *There are modification sets of density zero in* R.

THEOREM 2. *If* E *is a modification set in* R *then* $E\cap Z$ *is a modification set in* Z.

THEOREM 3. *There are modification sets of density zero in* Z.

REMARK. Modification sets can of course not be *too* small. For instance, every modification set in R has infinite measure (Plancherel); no lacunary set in Z is a modification set; no set of positive integers is a modification set (F. and M. Riesz). On the other hand, largeness is not enough: Theorem 2 shows that the complement of Z in R is not a modification set.

PROOF OF THEOREM 1. Choose integers $\lambda_1, \lambda_2, \lambda_3, \cdots$ so that $\lambda_1=10$, $\lambda_k\geqq 4\lambda_{k-1}$. Let A_k be the set of all numbers of the form

$$\pm\lambda_k+\epsilon_{k-1}\lambda_{k-1}+\cdots+\epsilon_1\lambda_1 \tag{1}$$

[1] Research supported by National Science Foundation Grant GP-6764.

where $\epsilon_i = 1$ or 0 or -1, let B_k be the union of all intervals of length $2k$ whose centers are in A_k, and put $E = B_1 \cup B_2 \cup B_3 \cup \cdots$.

Given $t > 10$, let $k = k(t)$ be the largest integer such that $\lambda_k \leqq 2t$. Then $E \cap [-t, t] \subset B_1 \cup \cdots \cup B_k$. Since A_i has $2 \cdot 3^{i-1}$ points, $m(B_i) \leqq 4i \cdot 3^{i-1}$. Hence

$$\frac{m(E \cap [-t, t])}{2t} \leqq \frac{1}{2t} \sum_{i=1}^{k} m(B_i) < \frac{2k \cdot 3^k}{\lambda_k} \leqq \frac{4k}{5} \cdot \left(\frac{3}{4}\right)^k$$

which tends to 0 as t (and hence k) tends to ∞. Thus E has density zero.

For $k = 1, 2, 3, \cdots$, let σ_k be the measure on T whose Fourier series is the formal expansion of the Riesz product

$$d\sigma_k(x) \sim \prod_{j=k}^{\infty} (1 + \cos \lambda_j x). \tag{2}$$

Then σ_k is a bounded, positive, continuous, and singular measure on T [2, p. 209] and $\hat{\sigma}_k(n) = 0$ unless $n \in \{0\} \cup A_k \cup A_{k+1} \cup \cdots$.

Now choose $f \in L^1(R)$ so that $\hat{f}$ has compact support and two continuous derivatives. Then $x^2 f(x)$ is bounded, so that $\sum |f(x - 2\pi j)|$, $j \in Z$, is a continuous periodic function. Fix k so that $\hat{f}(t) = 0$ whenever $|t| > k$. If we average the left side of (3) below over $-\pi \leqq s \leqq \pi$, and apply Fubini's theorem, we see that there exists an s (fixed from now on) such that

$$\int_{-\pi}^{\pi} \sum_{j=-\infty}^{\infty} |f(x - s - 2\pi j)| \, d\sigma_k(x) \leqq \int_{-\infty}^{\infty} |f(y)| \, dy = \|f\|_1. \tag{3}$$

Define a measure μ on R by requiring that

$$\int_{-\infty}^{\infty} g \, d\mu = \int_{-\pi}^{\pi} \sum_{j=-\infty}^{\infty} g(x - s - 2\pi j) f(x - s - 2\pi j) \, d\sigma_k(x) \tag{4}$$

for every bounded continuous g. Then μ is a singular measure on R whose total variation satisfies $\|\mu\| \leqq \|f\|_1$, by (3). The Poisson summation formula now gives

$$\int_{-\infty}^{\infty} e^{-itx} d\mu(x) = \int_{-\pi}^{\pi} \sum_{j=-\infty}^{\infty} e^{-it(x-s-2\pi j)} f(x - s - 2\pi j) \, d\sigma_k(x)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \hat{f}(t - n) e^{-in(x-s)} d\sigma_k(x) = \sum_{n=-\infty}^{\infty} \hat{\sigma}(n) \hat{f}(t - n) e^{ins}$$

which is the same as

$$\hat{\mu}(t) = \hat{f}(t) + \sum_{n\neq 0} \hat{\sigma}(n)\hat{f}(t-n)e^{ins} \qquad (t \in R). \tag{5}$$

In the last sum, $\hat{\sigma}(n)=0$ unless $n \in A_k \cup A_{k+1} \cup \cdots$, and $\hat{f}(t-n)=0$ if $|t-n| \geqq k$. Hence $\hat{\mu}(t)=\hat{f}(t)$ except possibly in $B_k \cup B_{k+1} \cup B_{k+2} \cup \cdots$ which is a subset of E.

To conclude the proof, let f be an arbitrary member of $L^1(R)$. Then $f=\sum f_n$ where $\sum \|f_n\|_1 < \infty$ and each $\hat{f}_n$ has compact support and two continuous derivatives. The preceding step shows that there are singular measures μ_n with $\|\mu_n\| \leqq \|f_n\|_1$, such that $\hat{\mu}_n(t)=\hat{f}_n(t)$ outside E. The series $\sum \mu_n$ then converges in the total variation norm to a measure μ which is therefore also singular, and if t is not in E we have

$$\hat{\mu}(t) = \sum \hat{\mu}_n(t) = \sum \hat{f}_n(t) = \hat{f}(t). \tag{6}$$

Thus E is a modification set in R.

Proof of Theorem 2. Let E be a modification set in R. Choose $f \in L^1(T)$, regard f as a member of $L^1(R)$ which vanishes outside $[-\pi, \pi)$, and let μ be a singular measure on R such that $\hat{\mu}(t)=\hat{f}(t)$ outside E. For $V \subset [-\pi, \pi)$, define $\sigma(V) = \sum \mu(V-2\pi j)$, $j \in Z$. Then σ is a singular measure on T, and $\hat{\sigma}(n)=\hat{\mu}(n)$ for every $n \in Z$. If $n \in Z$ and $\hat{f}(n) \neq \hat{\sigma}(n)$ it follows that $n \in E \cap Z$. So $E \cap Z$ is a modification set in Z.

Proof of Theorem 3. If E is one of the sets constructed in the proof of Theorem 1 then $E \cap Z$ has density zero in Z. Hence Theorem 3 follows from Theorem 2.

References

1. Walter Rudin, *Modifications of Fourier transforms*, Proc. Amer. Math. Soc. (to appear).

2. Antoni Zygmund, *Trigonometric series*, 2nd ed., Vol. I, Cambridge Univ. Press, New York, 1959.

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