

SPHERICAL FUNCTIONS ON A p -ADIC CHEVALLEY GROUP

BY I. G. MACDONALD

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1. Introduction. In [6] Satake set up a general theory of zonal spherical functions on a reductive linear algebraic group over a p -adic field. One problem left open was the determination of the explicit form of the spherical functions and the Plancherel measure. In this note we shall present explicit formulas for these things in the case of a Chevalley group. Detailed proofs will appear elsewhere.

2. Notation. Let \mathfrak{o} be a complete discrete valuation ring and K its field of fractions. Choose a generator π of the maximal ideal of \mathfrak{o} . Suppose that the residue field of \mathfrak{o} is *finite* and has q elements. Then K is locally compact (and totally disconnected) with respect to the valuation topology.

Let \mathfrak{g} be a complex semi-simple Lie algebra, \mathfrak{h} a Cartan sub-algebra of \mathfrak{g} , and V the vector-space dual of \mathfrak{h} . The dual V^* of V is then identified with \mathfrak{h} . If $\xi \in \mathfrak{h} = V^*$ and $\eta \in V$ we write (ξ, η) for the value of η at ξ (or of ξ at η). Let $\Delta \subset V$ be the set of (nonzero) roots of \mathfrak{g} relative to \mathfrak{h} , and for each root α let $\alpha^* \in V^*$ be the corresponding coweight. Choose a system of fundamental roots Π in Δ ; we can then speak of positive and negative roots. The roots generate a free abelian group $R \subset V$ of rank $l = \dim V$, and Π is a basis of R . Let R^* denote the set of all $\xi \in V^*$ such that $(\xi, \alpha) \in \mathbb{Z}$ for all roots α . Then $R^* \cong \text{Hom}(R, \mathbb{Z})$ and is a free abelian group of rank l contained in V^* , and contains all the coweights α^* . Let R_+^* be the set of all $\lambda \in R^*$ such that $(\lambda, \alpha) \geq 0$ for all positive roots α .

The Weyl group W of \mathfrak{g} (relative to \mathfrak{h}) acts on \mathfrak{h} and hence on V by transposition. Any element of R^* can be brought into R_+^* by the action of some element of W . For each $w \in W$ let $n(w)$ denote the number of positive roots α such that $w(\alpha) < 0$, and let $P(t)$ denote the Poincaré polynomial

$$(2.1) \quad P(t) = \sum t^{n(w)}$$

summed over all $w \in W$. More generally, if $\lambda \in R^*$ let

$$(2.2) \quad P_\lambda(t) = \sum t^{n(w)}$$

summed over all $w \in W$ such that $w(\lambda) = \lambda$. Then $P_\lambda(t)$ divides $P(t)$, and we put

$$(2.3) \quad P^\lambda(t) = P(t)/P_\lambda(t).$$

Let X_α ($\alpha \in \Delta$); α^* ($\alpha \in \Pi$) be a Chevalley basis of \mathfrak{g} relative to \mathfrak{h} . Let \mathfrak{g}_Z be the Lie sub-ring of \mathfrak{g} generated by these basis elements, and let $\mathfrak{g}_K = K \otimes \mathfrak{g}_Z$ and $\mathfrak{g}_\mathfrak{o} = \mathfrak{o} \otimes \mathfrak{g}_Z$. For each $t \in K$ and $\alpha \in \Delta$ let $x_\alpha(t) = \exp(t \operatorname{ad} X_\alpha)$, which is an automorphism of \mathfrak{g}_K . For each homomorphism $\chi: R \rightarrow K^*$ let $h(\chi)$ be the automorphism of \mathfrak{g}_K which leaves each basis element α^* fixed and maps X_α to $\chi(\alpha)X_\alpha$ for all $\alpha \in \Delta$. Let G be the group of automorphisms of \mathfrak{g}_K generated by the $x_\alpha(t)$ and the $h(\chi)$.

We define sub-groups U, H, N of G as follows. H is generated by the $h(\chi)$ and is isomorphic to $\operatorname{Hom}(R, K^*)$; N is generated by the $x_\alpha(t)$ for all $t \in K$ and all positive roots α ; and U is the stabilizer in G of the lattice $\mathfrak{g}_\mathfrak{o}$. The group G is locally compact and U is a maximal compact sub-group of G , and is open in G .

The choice of generator π enables us to embed Z in K^* by $z \mapsto \pi^z$, and hence $R^* \cong \operatorname{Hom}(R, Z)$ in $H \cong \operatorname{Hom}(R, K^*)$. If $\lambda \in R^*$, let π^λ denote the image of λ in H ; explicitly, π^λ maps each X_α to $\pi^{\langle \lambda, \alpha \rangle} X_\alpha$. Finally, let H^0 (resp. H^0_+) be the image of R^* (resp. R^*_+) in H . Then [1], [4]:

(2.4) *Cartan decomposition.* $G = UH^0_+U$, and G is the disjoint union of the double cosets $U\pi^\lambda U$ where $\lambda \in R^*_+$.

(2.5) *Iwasawa decomposition.* $G = UH^0N$, and G is the disjoint union of the double cosets $U\pi^\lambda N$ where $\lambda \in R^*$.

3. Spherical functions. In this paragraph G is any unimodular separable locally compact topological group, and U is a compact sub-group of G . Let $L(G, U)$ be the \mathbb{C} -vector space of all complex-valued continuous functions on G which have compact support and are bi-invariant with respect to U , that is to say are constant on each double coset UxU of U in G . Fix a Haar measure dx on G and define multiplication in $L(G, U)$ to be convolution:

$$(f * g)(x) = \int_G f(xy^{-1})g(y)dy.$$

Then $L(G, U)$ is a \mathbb{C} -algebra. Assume now that $L(G, U)$ is *commutative*. Then a *zonal spherical function* (z.s.f. for short) on G relative to U is defined to be a complex-valued continuous function ω on G which satisfies the following three conditions: (1) ω is bi-invariant with respect to U ; (2) $\omega(1) = 1$; (3) $f * \omega = \lambda_f \omega$ for all $f \in L(G, U)$, where λ_f is a complex number depending on f . For the theory of spherical functions we refer to [2], [3], [7].

A z.s.f. ω is said to be of *positive type* if

$$\int_G \int_G \omega(xy^{-1})\phi(x)\overline{\phi(y)}dx dy \geq 0$$

for all continuous functions $\phi: G \rightarrow \mathbb{C}$ with compact support. Let Ω (resp. Ω^+) denote the set of all z.s.f. (resp. z.s.f. of positive type) on G relative to U . For each $f \in L(G, U)$ the *Fourier transform* of f is the function on Ω defined by

$$\hat{f}(\omega) = \int_G f(x)\omega(x^{-1})dx.$$

The basic fact [2] is that there exists a unique positive measure $d\omega$ on Ω^+ , called the *Plancherel measure*, such that

$$\int_G |f(x)|^2 dx = \int_{\Omega^+} |\hat{f}(\omega)|^2 d\omega$$

for all $f \in L(G, U)$.

4. Spherical functions on a Chevalley group. Now let G and U be as defined in §2. An elementary argument, based on the existence of an automorphism of \mathfrak{g} which sends every coweight α^* to its negative, shows that $L(G, U)$ is commutative (see [3, p. 408] for the corresponding result in the real case).

The following construction gives z.s.f.'s on G (relative to U). Let $\phi: H^0 \rightarrow \mathbb{C}^*$ be a homomorphism, and use the Iwasawa decomposition (2.5) to extend ϕ to a function on the whole of G : if $x \in Uhn$ with $h \in H^0$, define $\phi(x)$ to be $\phi(h)$. Now let

$$(4.1) \quad \omega_\phi(x) = \int_U \phi(x^{-1}u)du$$

where du is Haar measure on the compact group U , normalized so that the measure of U is 1. Then ω_ϕ is a z.s.f. on G relative to U , and moreover *all* z.s.f. are obtained in this way [6].

It is convenient to parametrize the ω_ϕ as follows [6]. Given $\phi: H^0 \rightarrow \mathbb{C}^*$, there exists $s_0 \in V$ such that $\phi(\pi^\lambda) = q^{-(\lambda, s_0)}$ for all $\lambda \in R^*$; this s_0 is not uniquely determined by ϕ , but is unique modulo $(2\pi i/\log q)R$. Let

$$(4.2) \quad \sigma = \frac{1}{2} (\text{sum of positive roots})$$

and let $s = s_0 + \sigma$. Then we take $s \in V$ as parameter for the z.s.f. ω_ϕ , and we write ω_s in place of ω_ϕ . The advantage of this parametrization

is that $\omega_s = \omega_{s'}$ if and only if $s' \equiv ws \pmod{(2\pi i / \log q)R}$ for some $w \in W$. The integral (4.1) now becomes

$$(4.3) \quad \omega_s(x) = \int_U q^{\mu(x^{-1}u), \sigma^{-s}} du$$

where μ is the projection of G on $R^* = U \backslash G / N$.

Since ω_s is bi-invariant with respect to U , it follows from (2.4) that ω_s is uniquely determined by its values on H_+^0 .

For each $s \in V$ which does not lie on a root hyperplane, i.e. which is such that $(\alpha^*, s) \not\equiv 0 \pmod{(2\pi i / \log q)}$ for all $\alpha \in \Delta$, let

$$(4.4) \quad c(s) = \prod_{\alpha > 0} \frac{1 - q^{-1 - (\alpha^*, s)}}{1 - q^{- (\alpha^*, s)}}$$

Then we have the following formula.

THEOREM 1. *Let $\lambda \in R_+^*$ and let $s \in V$ be as above. Then*

$$(4.5) \quad \omega_s(\pi^\lambda) = \frac{1}{P(q^{-1})} \sum_{w \in W} q^{(\lambda, ws - \sigma)} c(ws)$$

where the polynomial P is that defined in (2.1).

This is proved by computing the integral (4.3). For this purpose the Iwahori decomposition of U [4] is used to split up U into a disjoint union of double cosets U_w ($w \in W$), and the integral over U_w is calculated by induction on the integer $n(w)$. (It is not the case that the terms in the sum on the right-hand side of (4.5) correspond to the integrals over the double cosets U_w ; it is more complicated than that.)

Formula (4.5) is still valid, suitably interpreted, if s lies on a root hyperplane in V . Let $A = \mathbb{Q}[R^*]$ be the rational group algebra of R^* and let e^λ denote the canonical image of $\lambda \in R^*$ in A . Consider the fraction

$$(4.6) \quad Q_\lambda = \sum_{w \in W} e^{-w\lambda} \prod_{\alpha > 0} \frac{1 - q^{-1} e^{w\alpha^*}}{1 - e^{w\alpha^*}}$$

where $\lambda \in R^*$. On the face of it, Q_λ is an element of the field of fractions of the integral domain A . In fact it is easy to see that $Q_\lambda \in A$. Now any homomorphism $R^* \rightarrow \mathbb{C}^*$ extends to a ring homomorphism $A \rightarrow \mathbb{C}$; if $\psi_s: A \rightarrow \mathbb{C}$ is the extension of $\lambda \mapsto q^{- (s, \lambda)}$, then (4.5) takes the form

$$(4.5') \quad \omega_s(\pi^\lambda) = \frac{q^{-\langle \lambda, \sigma \rangle}}{P(q^{-1})} \psi_s(Q_\lambda)$$

and is valid for all $s \in V$.

Theorem 1 can be reformulated as a purely algebraic statement about the structure of the ring $L(G, U)$. For each $\lambda \in R^*$, let χ_λ denote the characteristic function of the double coset $U\pi^\lambda U$ in G . The χ_λ with $\lambda \in R^*_\dagger$ form a basis of $L(G, U)$ as a \mathbb{C} -vector space. Their Fourier transforms $\hat{\chi}_\lambda$ are functions on the space Ω of z.s.f. and hence can be regarded as W -invariant functions on V . In fact [6, §6] they are polynomial functions and hence may be regarded as elements of $\mathcal{Q}[R^*]$, or of $\mathbb{C}[R^*]$. Thus Fourier transformation gives rise to a homomorphism $L(G, U) \rightarrow \mathbb{C}[R^*]$, and Theorem 1 then takes the following form:

THEOREM 1'. *Let $\mathbb{C}[R^*]^W$ be the sub-algebra of W -invariant elements of the group algebra $\mathbb{C}[R^*]$, and let $f: L(G, U) \rightarrow \mathbb{C}[R^*]^W$ be the \mathbb{C} -linear mapping defined by*

$$(4.7) \quad f(\chi_\lambda) = \frac{q^{\langle \lambda, \sigma \rangle}}{P^\lambda(q^{-1})} Q_\lambda,$$

where Q_λ is defined in (4.6), σ in (4.2) and P^λ in (2.3). Then f is a \mathbb{C} -algebra isomorphism.

5. Plancherel measure. Let Ω^+ be the space of z.s.f. of positive type on G relative to U . Let V_0 be the real vector subspace of V spanned by Δ , so that V is the complexification of V_0 . If $s \in iV_0$, then ω_s is a z.s.f. of positive type. The ω_s with $s \in iV_0$ form a subspace Ω_0^+ of Ω^+ ; in general Ω_0^+ is not the whole of Ω^+ (see [6, Appendix 2], for the case of $PGL(2, K)$), but we can afford to ignore the rest of Ω^+ because it will have Plancherel measure zero. The space Ω_0^+ is the quotient of iV_0 firstly by the lattice $(2\pi i/\log q)R$ and then by the Weyl group W , i.e. it is of the form T/W where T is an l -dimensional real torus. Let ds be a Euclidean measure on iV_0 and let dx be a Haar measure on G . Then

THEOREM 2. *The Plancherel measure $d\omega$ on Ω^+ is concentrated on Ω_0^+ . If ds and dx are suitably normalized, then*

$$(5.1) \quad d\omega_s = |c(s)|^{-2} ds \quad (s \in iV_0)$$

where $c(s)$ is defined in (4.4), so that

$$(5.2) \quad |c(s)|^{-2} = \prod_\alpha \frac{1 - q^{-(\alpha^*, s)}}{1 - q^{-1-(\alpha^*, s)}}$$

the product being taken over all roots α , positive and negative.

This is an easy consequence of Theorem 1. It agrees with Mautner's result [5] for the case of $PGL(2, K)$.

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OXFORD UNIVERSITY, ENGLAND