ANALOGUES OF ARTIN'S CONJECTURE[1]

BY LARRY JOEL GOLDSTEIN

Communicated by W. Feit, November 17, 1967

Artin's celebrated conjecture on primitive roots (Artin [1, p. viii], Hasse [2], Hooley [3]) suggests the following

Conjecture. Let S' be a set of rational primes. For each $q \in S$, let L_q be an algebraic number field of degree $n(q)$. For every square-free integer k, divisible only by primes of S, define L_k to be the composite of all L_q, $q \mid k$, and denote $n(k) = \deg(L_k/Q)$. Assume that $\sum_k 1/n(k)$ converges, where the sum is over those k for which L_k is defined. Then the natural density of the set P of all primes p which do not split completely in each L_q exists and has the value $\sum_k \mu(k)/n(k)$, where μ is the Möbius function and the term $k=1$ has been included with $n(1)=1$.

If $S = \{\text{all rational primes}\}$, $L_q = Q(\zeta_q, a^{1/q})$, $a \in Z$, $\zeta_q =$ a primitive qth root of 1, then the conjecture is equivalent to Artin's conjecture. If S is a finite set, then the conjecture is easily verifiable using the prime ideal theorem. For $S = \{\text{all rational primes}\}$, $L_q = Q(\zeta_{q^r})$, the conjecture has been proved by Knobloch [4] (for $r=2$ and only for Dirichlet densities) and by Mirsky [5].

We have proved the following theorems, whose proofs will appear elsewhere.

THEOREM 1. *Let there exist a finite set $S_0 \subset S$ such that $L_q \supset Q(\zeta_{q^2})$ for $q \in S - S_0$, and L_q/Q is normal for all $q \in S$. Then the conjecture is true.*

THEOREM 2. *Suppose that for each finite subset $S_0 \subset S$ there exists a family of algebraic number fields $\{L_q'\}_{q \in S}$ such that*

(1) $L_q = L_q'$ for $q \in S_0$,

(2) $L_q' \subset L_q$ for all $q \in S$,

(3) $L_q' \neq Q$ for all $q \in S$,

(4) the conjecture is true for $\{L_q'\}_{q \in S}$.

Then the conjecture is true for $\{L_q\}_{q \in S}$.

THEOREM 3. *If the density $d(P)$ of P exists, then*

$$d(P) \leqq \sum_k \mu(k)/n(k).$$

[1] Research partially supported by Air Force Office of Scientific Research Grant No. SAR/F-44620-67.

Theorem 1 is the main result. Theorems 2 and 3 are elementary in character. The proof of Theorem 1 is divided into two parts: First, it is shown that one may compute the number of primes $p \leqq x$ in P by computing the number of primes $p \leqq x$ which do not split completely in L_q for all "sufficiently small q" where the upper bound for q is a function of x. Computing this latter quantity is reduced to computing the number of prime ideals of L_k which have norm $\leqq x$, for all "sufficiently small k." The prime ideal theorem asserts that this latter quantity is asymptotically equal to $x/\log x$. But the error term will, in general, depend on L_k. The second part of the proof consists in showing that by restricting k to be "sufficiently small" one can choose the error term to be independent of k. This result constitutes a generalization of the uniform prime number theorem of Siegel and Walfisz (Prachar [6, p. 144]) for primes in arithmetic progressions. In fact, we can prove our theorem in a very general setting, which, although not required for the proofs of Theorems 1–3, seems interesting for its own sake.

Let K be a normal algebraic number field of degree n and discriminant d. Let $\alpha \to \alpha^{(j)}$ $(1 \leqq j \leqq n)$ be the embeddings of K in the complex numbers $\mathbf{C}$, ordered so that the first r_1 are real and the jth and $(j+r_2)$th $(r_1+1 \leqq j \leqq r_1+r_2)$ constitute a pair of complex-conjugate embeddings. Let

$$\begin{aligned} n_j &= 1, \quad 1 \leqq j \leqq r_1 \\ &= 2, \quad r_1 + 1 \leqq r_1 + r_2. \end{aligned}$$

For $\alpha \in K^* = K - \{0\}$, let $\alpha \equiv 1 \pmod{^* \mathfrak{a}}$ mean that α is multiplicatively congruent to 1 modulo the K-ideal $\mathfrak{a}$. For $\alpha \in K^*$, denote by (α) the K-ideal generated by α. Let χ be a grossencharacter of K having conductor $\mathfrak{f}$. For $\alpha \equiv 1 \pmod{^* \mathfrak{f}}$, let

$$\chi((\alpha)) = \prod_{j=1}^{r_1+r_2} \left(\frac{\alpha^{(j)}}{|\alpha^{(j)}|} \right)^{m_j} |\alpha^{(j)}|^{i n_j \phi_j}$$

where $m_j = 0, 1$ and $\phi_j \in R$ are normalized so that $\sum_{j=1}^{r_1+r_2} n_j \phi_j = 0$. Let

$$\pi(x, K, \chi) = \sum_{N\mathfrak{p} \leqq x;\, (\mathfrak{p}, \mathfrak{f}) = 1} \chi(\mathfrak{p})$$

where the sum is over primes $\mathfrak{p}$ of K. For $A > 0$, define $B(A) = \{\chi$ a grossencharacter of $K \mid |\phi_j| \leqq A,\ 1 \leqq j \leqq r_1 + r_2\}$. Then we have the following generalization of the Siegel-Walfisz theorem:

THEOREM 4. *Let $A > 0$, $\epsilon > 0$ be given. Then there exists a positive constant $c = c(A, \epsilon)$, not depending on K, n, d, or χ such that for $\chi \in B(A)$,*

$$\pi(x, K, \chi) = E(\chi) \operatorname{li} x + O(Dx \log^2 x \exp\{-cn(\log x)^{1/2}/D\}), \quad x \to \infty$$

where the 0-*term constant does not depend on* K, χ, n *or* d *and*

$$E(\chi) = 0, \quad \chi \neq \textit{the trivial grossencharacter}$$
$$= 1, \quad \chi = \textit{the trivial grossencharacter},$$
$$\operatorname{li} x = \int_2^x \frac{dy}{\log y},$$
$$D = n^4[|d| N(\mathfrak{f})]^\epsilon c^{-n}.$$

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Yale University