NEW SIMPLE LIE ALGEBRAS OF TYPE D_4

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This brief note is to demonstrate the existence of a new class of (exceptional) Lie algebras of type D_4. The construction stems from a cyclic sixth degree extension P/Φ, together with an element γ of norm 1 in the unique cubic subfield F/Φ of P/Φ, where $\gamma \notin N_{P/F}(P^*)$. Each such γ will determine a non-Jordan (see [1] for definition) Lie algebra $\mathfrak{L}(\gamma)$, of type $D_{4\mathrm{III}}$. Two algebras of this form, $\mathfrak{L}(\gamma)$ and $\mathfrak{L}(\rho)$, will be isomorphic if and only if γ differs from a conjugate of ρ by a norm in $N_{P/F}(P^*)$. The possibility of obtaining new D_4's from such a construction was first conjectured in [2].

We shall make free use of the well-known theory of finite Galois descent for nonassociative algebras and all the results which we use may be found in ([5, Chapter 10]).

0. Preliminaries. We assume without further mention that all fields which appear here have characteristic unequal to 2 or 3.

Let $\mathfrak{J}$ be a split exceptional central simple Jordan algebra over P, $\{e_1, e_2, e_3\}$ a set of supplementary orthogonal primitive idempotents and let $\mathfrak{D} = \mathfrak{D}(\mathfrak{J}/\Sigma Pe_i)$ be the subalgebra of the derivation algebra of $\mathfrak{J}$ annihilating ΣPe_i. Then $\mathfrak{D}$ is the split D_4. If $\mathfrak{L}$ is a Φ-algebra form of $\mathfrak{D}$ $(P \supset \Phi)$, then we let $\mathfrak{L}^*$ be the Φ-subalgebra of $\mathrm{End}_P \mathfrak{J}$ generated by $\mathfrak{L}$ (we view $\mathfrak{L}$ as a Φ-subspace of $\mathfrak{D}$ which contains a Φ-basis which is also a P-basis for $\mathfrak{D}$). It is known that $(\mathfrak{L}^*)_P \cong P_8 \oplus P_8 \oplus P_8$. $\mathfrak{L}$ is special (i.e., has the form $\mathfrak{s}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a central simple associative algebra of degree 8 with involution) if and only if $\mathfrak{L}^*$ has proper ideals. When $\mathfrak{L}^*$ is simple, i.e. when $\mathfrak{L}$ is exceptional, then $\mathfrak{L}$ is of known type—a Jordan D_4—if and only if $\mathfrak{L}^*$ is a total matrix algebra over its center. $\mathfrak{L}$ is of type $D_{4\mathrm{III}}$ $(D_{4\mathrm{VI}})$ if the center of $\mathfrak{L}^*$ is a cyclic (noncyclic) extension of Φ.

If $\mathfrak{L}$ is of type $D_{4\mathrm{III}}$ and F is the center of $\mathfrak{L}^*$—the canonical $D_{4\mathrm{I}}$-field extension of $\mathfrak{L}$—then $\mathfrak{L}$ is a non-Jordan $D_{4\mathrm{III}}$ if and only if none of the simple components of $(\mathfrak{L}^*)_F$ is a total matrix algebra.

We shall need some technical information about the structure of split Cayley algebra. For this we refer to [6] and for convenience list the results below for reference.

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Let $\mathfrak{C}$ be the vector space of all 2×2 matrices.

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$$

where $\alpha, \beta \in P$ and $a, b \in P^{(3)} = P\times P\times P$. $\mathfrak{C}$ is equipped with a bilinear multiplication and an involution, which are defined by

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + a\cdot d & \alpha c + \delta a - b\wedge d \\ \gamma b + \beta d + a\wedge c & \beta\delta + b\cdot c \end{pmatrix}$$

and

$$\overline{\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}} = \begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix}$$

where $a\cdot d$ and $b\wedge d$ denote the usual dot and cross product in $P^{(3)}$. $\mathfrak{C}$ is a split Cayley algebra over P.

The mapping $x\to x\bar{x} = n(x) \in P \subset \mathfrak{C}$ is a nondegenerate quadratic form of maximal Witt index, the generic norm on $\mathfrak{C}$. If $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ is the usual cartesian basis for $P^{(3)}$ and we define $u_i \in \mathfrak{C}$ by

$$u_i = \begin{pmatrix} 0 & 0 \\ \epsilon_i & 0 \end{pmatrix} \qquad u_{i+4} = -2\begin{pmatrix} 0 & \epsilon_i \\ 0 & 0 \end{pmatrix} \qquad 1 \leqq i \leqq 3$$

$$u_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad u_8 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

then $u_1, \cdots, u_8$ is a basis for $\mathfrak{C}$ where $n(u_i, u_j) = \delta_{i+4,j}$, $n(x, y)$ denoting the norm bilinear form of $n(x)$ with $i+4$ taken modulo 8.

The multiplication table for $\mathfrak{C}$, which is given in [6] for this basis, will be invaluable. (See p. 483.)

The product u_iu_j is found in the ith-row, jth-column.

1. Lemma 1. *Let $\mathfrak{C}$ be the split Cayley algebra over P and let $\gamma_1, \gamma_2, \gamma_3 \in P^*$ with $\gamma_1\gamma_2\gamma_3 \in (P^*)^2$. Then there exists a related triple (T_1, T_2, T_3) of proper similarities in $\mathfrak{C}$ where each T_i is selfadjoint with ratio γ_i. In particular, $(T_i)^2 = (\gamma_1, \gamma_2, \gamma_3)$.*

Proof. (See [4] for the definition of related triples.) We take $\mathfrak{C}$ as in the preceding section and assume without loss of generality that $\gamma_1\gamma_2\gamma_3 = 1$. Let T_i, $i = 1, 2, 3$, be the linear transformation whose matrix T_i, with respect to $u_1, \cdots, u_8$, is given on the following page.

$$T_1 = \left[\begin{array}{cccc|cccc} & & & & \frac{1}{2}\gamma_1 & & & \\ & 0 & & & & \frac{1}{2}\gamma_1 & & \\ & & & & & & \frac{1}{2}\gamma_1 & \\ & & & & & & & \frac{1}{2}\gamma_3^{-1} \\ \hline 2 & & & & & & & \\ & 2 & & & & 0 & & \\ & & 2 & & & & & \\ & & & 2\gamma_2^{-1} & & & & \end{array}\right]$$

$$T_2 = \left[\begin{array}{cccc|cccc} & & & & \frac{1}{2}\gamma_2 & & & \\ & 0 & & & & \frac{1}{2}\gamma_2 & & \\ & & & & & & \frac{1}{2}\gamma_2 & \\ & & & & & & & \frac{1}{2}\gamma_1^{-1} \\ \hline 2 & & & & & & & \\ & 2 & & & & 0 & & \\ & & 2 & & & & & \\ & & & 2\gamma_3^{-1} & & & & \end{array}\right]$$

$$T_3 = \left[\begin{array}{cccc|cccc} & & & & \frac{1}{2}\gamma_3 & & & \\ & 0 & & & & \frac{1}{2}\gamma_3 & & \\ & & & & & & \frac{1}{2}\gamma_3 & \\ & & & & & & & \frac{1}{2}\gamma_2^{-1} \\ \hline 2 & & & & & & & \\ & 2 & & & & 0 & & \\ & & 2 & & & & & \\ & & & 2\gamma_1^{-1} & & & & \end{array}\right].$$

Using the fact that $\{u_1, \cdots, u_4\}$, $\{u_5, \cdots, u_8\}$ span supplementary totally isotropic subspaces, it is easy to see that $n(u_i, u_jT_k)=0$ if $i\neq j$. This implies that each T_i is selfadjoint. We have

$$n(u_iT_k, u_jT_k) = 0 = n(u_i, u_j) \quad \text{if } j \neq i+4 \pmod 8$$

$$n(u_iT_k, u_{i+4}T_k) = \gamma_k n(u_i, u_{i+4}) \quad 1 \leqq i \leqq 3$$

$$n(u_4T_k, u_8T_k) = \gamma_{2+k}^{-1}\gamma_{1+k}^{-1} n(u_4, u_8) = \gamma_k n(u_4, u_8) \quad \text{where the}$$

subscripts on the γ's are taken modulo 3.

This shows that T_k is a similarity of ratio γ_k. To complete the proof we must show that

$$\overline{u_i u_j T_1} = \gamma_1(u_i T_2)(u_j T_3) \quad \text{for all } i, j.$$

By examining the multiplication table for the u's, we see that $u_i u_j = 0$ implies $(u_i T_2)(u_j T_3) = 0$. The remaining 32 cases are verified by straightforward computations. q.e.d.

The construction. Let P/Φ be a cyclic sixth degree Galois extension with F/Φ the cubic subfield of P/Φ, and let s be a generator for gal (P/Φ). Choose $\gamma \in F^*$ with $1 = N_{F/\Phi}(\gamma) = \gamma\gamma^s\gamma^{s^2}$.

Take $\mathfrak{C}_0$ as the split Cayley algebra over Φ with basis $\{u_1, \cdots, u_8\}$ as described in §0 and let $\mathfrak{C} = \mathfrak{C}_{0P}$ be the split Cayley algebra over P with basis $\{u_1, \cdots, u_8\}$. We let $T = (T_1, T_2, T_3)$ be the related triple of similarities in $\mathfrak{C}$ constructed in Lemma 1 with the ingredients $\gamma, \gamma^{s^2}, \gamma^s$. Finally let S be the s-linear automorphism of $\mathfrak{C}$ which fixes $\mathfrak{C}_0$. Let $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3)$ and let $[(123), S]$ and $[1, T]$ be the transformations in $\Gamma L_\Phi(\mathfrak{J}/\Sigma Pe_i)$ as defined in [1, Equations 9 and 5].

LEMMA 2. $[(123), S][1, T] = [1, T][(123), S]$.

PROOF. We must show that $ST_1 = T_2S$, $ST_2 = T_3S$ and $ST_3 = T_1S$. If we define $(\alpha_{ij})^s = (\alpha_{ij}^s)$ in P_8, and if $\boldsymbol{T}_i$ denotes the matrix of T_i with respect to $\{u_1, \cdots, u_8\}$, then our conditions reduce to $\boldsymbol{T}_1 = \boldsymbol{T}_2^s$, $\boldsymbol{T}_2 = \boldsymbol{T}_3^s$ and $\boldsymbol{T}_3 = \boldsymbol{T}_1^s$. But this is immediate from the form of $\boldsymbol{T}_i$ given in Lemma 1. q.e.d.

Assume now that $\gamma \notin N_{P/F}(P^*)$ (this assumption is nonvacuous over finite algebraic number fields) and let $C(\gamma)$ be the transformation $[(123), S][1T]$ in $\mathfrak{h}(\mathfrak{C}_3)$. $C(\gamma)$ is s-linear and $C(\gamma)^6 = (\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_1 = \gamma^3$, $\alpha_2 = (\gamma^{s^2})^3$ and $\alpha_3 = (\gamma^s)^3$. It follows from this that conjugation by $C(\gamma)$ induces a pre-cocycle of gal(P/Φ) in Aut$_\Phi\mathfrak{D}$, $\mathfrak{D} = \mathfrak{D}(\mathfrak{J}/\Sigma Pe_i)$, and hence fixes a Φ-form, say $\mathfrak{L}(\gamma)$. $\mathfrak{L}(\gamma)$ is clearly of type $D_{4\text{III}}$ with F/Φ as its canonical $D_{4\text{I}}$-field extension. Since the division algebra parts of the simple ideals of $\mathfrak{L}^*(\gamma)_F$ are the cyclic algebras $(P/F, \gamma^3)$, $(P/F, (\gamma^{s^2})^3)$, $(P/F, (\gamma^s)^3)$ and $\gamma^3 \notin N_{P/F}(P^*)$, we see that $\mathfrak{L}(\gamma)$ is a non-Jordan $D_{4\text{III}}$. Observe that the algebra $\mathfrak{L}(\gamma)$ is a twist of a Steinberg $D_{4\text{III}}$ and that this is precisely the situation conjectured at the end of [2].

Isomorphism conditions. Let P/Φ be as above. For any $\gamma \in F^*$ with $\gamma\gamma^s\gamma^{s^2} = 1$, we can define the algebra $\mathfrak{L}(\gamma)$ as in the preceding. Writing down explicitly the condition for isomorphism between $\mathfrak{L}(\gamma)$ and $\mathfrak{L}(\rho)$ we obtain (in terms of descent)

$$A^{-1}C(\gamma)A(\mu_1, \mu_2, \mu_3) = C(\rho) \qquad A \in GL(\mathfrak{J}/\Sigma Pe_i), \quad \mu_i \in P^*.$$

In particular we see that $[(123), 1]C(\gamma)[(132), 1] = C(\gamma^s)$ so $\mathfrak{L}(\gamma) \cong \mathfrak{L}(\gamma^s) \cong \mathfrak{L}(\gamma^{s^2})$. More generally we have

THEOREM. *Let P/Φ be cyclic sixth degree with F/Φ the cubic subextension. If γ, ρ are elements of F of norm 1, then $\mathfrak{L}(\gamma) \cong \mathfrak{L}(\rho)$ if and only if $\mathfrak{L}(\gamma)^*_F \cong \mathfrak{L}(\rho)^*_F$ (as algebras without involution).*

PROOF. One direction is clear. For the other, the condition $\mathfrak{L}(\gamma)^*_F \cong \mathfrak{L}(\rho)^*_F$ is equivalent to a relation of the form $\rho = \gamma^{s^{si}}\lambda\lambda^{s^3}$ for some i, $0 \leqq i \leqq 2$. The preceding discussion enables us to assume that $i = 0$, i.e. that $\rho = \gamma\lambda\lambda^{s^3}$. Observe that $N_{P/\Phi}(\lambda) = 1$ (take $N_{F/\Phi}$ of both sides) and set $\epsilon = \gamma(\lambda^{s^4}\lambda^{s^5})^{-1}$. Then $\epsilon\epsilon^{s^2}\epsilon^{s^4} = 1$ and we let E be the related triple described by Lemma 1 for ϵ, ϵ^{s^2}, ϵ^{s^4}. A straightforward calculation shows that

$$[1, E]^{-1}C(\gamma)[1, E]((\lambda^s\lambda^{s^2})^{-1}, (\lambda^{s^3}\lambda^{s^4})^{-1}, (\lambda\lambda^{s^5})^{-1}) = C(\rho)$$

so $\mathfrak{L}(\gamma) \cong \mathfrak{L}(\rho)$. q.e.d.

2. **Special fields.** As remarked above, our construction may be carried out over finite algebraic number fields. The results of [2] show that any $D_{4\text{III}}$ over such a field is split by a cyclic sixth degree extension P/Φ and by a slight modification of the proof of Proposition 3 of [2] we may assume that P has no real primes. Let $\mathfrak{L}$ be a non-Jordan $D_{4\text{III}}$ over Φ. Let $\mathfrak{L}_F$ (F as before) be the canonical $D_{4\text{I}}$ extension of $\mathfrak{L}$. In the indicated reference it is also shown that $\mathfrak{L}_F$ is fixed under conjugation by a semilinear transformation $[1, (C_i)]$ where $[1, (C_i)]^2 = (\gamma^3, (\gamma^{s^2})^3, (\gamma^{s3}))$, $\gamma \in F$, $N_{F/\Phi}(\gamma) = 1$. Since $\mathfrak{L}$ is a non-Jordan $D_{4\text{III}}$, $\gamma \notin N_{P/F}(P^*)$.

Let $\Delta = (P, t, \gamma^3)$, $t = s^3$. Then $\mathfrak{L}_F$ has a realization as $\mathfrak{s}(\Delta_4, J)$ which we can describe explicitly as follows:

Write $\Delta = P + CP$, $C^2 = \gamma^3$, $\alpha C = C\alpha^t$, define a Δ-module structure on $\mathfrak{C}$ by setting $x \cdot (\alpha + C\beta) = x\alpha + (xC_1)\beta$, and let $-$ denote the involution $\alpha + C\beta \to \alpha + C\beta^t$ in Δ. It follows from [3] that $\mathfrak{L}_F$ is isomorphic to the Lie algebra of all Δ-linear transformations in $\mathfrak{C}$ which are skew with respect to the nondegenerate-hermitian form

$$f(x, y) = n(x, y) + Cn(x, yC_1^{-1})^t$$

on $\mathfrak{C}/\Delta$. In case $\mathfrak{L} = \mathfrak{L}(\gamma)$, then $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for $\mathfrak{C}/\Delta$ and we compute

$$f(u_i, u_i) = C\tfrac{1}{2} \qquad 1 \leqq i \leqq 3$$

$$f(u_4, u_4) = C\tfrac{1}{2}\gamma^{s^2}.$$

In a forthcoming paper, the first author has shown that f cannot have maximal Witt index. However, using the Hasse principle for hermitian forms of type D we conclude that f has Witt index 0 if and only if there is a real prime p on F with $\mathfrak{L}_{F_p}$ the compact real D_4.

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_1	0	$\frac{1}{2}u_7$	$-\frac{1}{2}u_6$	u_1	$-u_8$	0	0	0
u_2	$-\frac{1}{2}u_7$	0	$\frac{1}{2}u_5$	u_2	0	$-u_8$	0	0
u_3	$\frac{1}{2}u_6$	$-\frac{1}{2}u_5$	0	u_3	0	0	$-u_8$	0
u_4	0	0	0	u_4	u_5	u_6	u_7	0
u_5	$-2u_4$	0	0	0	0	$4u_3$	$-4u_2$	$2u_5$
u_6	0	$-2u_4$	0	0	$-4u_3$	0	$4u_1$	$2u_6$
u_7	0	0	$-2u_4$	0	$4u_2$	$-4u_1$	0	$2u_7$
u_8	$2u_1$	$2u_2$	$2u_3$	0	0	0	0	$2u_8$

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