GROUPS OF DIMENSION 1 ARE LOCALLY FREE

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Our result is slightly more general than

THEOREM 1. *A torsion-free, finitely presented group G, with infinitely many ends, can be written as a nontrivial free product* $G_1 * G_2$.

The condition "finitely presented" can be weakened to: There is a finite complex K and a regular covering space $\tilde{K}$ with $H^1(\tilde{K})=0$, such that G is isomorphic to the group of covering translations of $\tilde{K}$.

From this we deduce

THEOREM 2. *If a finitely generated group G has cohomological dimension* $\leqq 1$, *then G is free* [1].

This is another way of stating the title theorem. Another consequence is

THEOREM 3. *If a finitely generated group G has a free subgroup of finite index, and if G is torsion-free, then G is free* [3].

(The references are to papers where these results have been conjectured.)

We shall indicate briefly how to prove Theorems 1 and 2. Details will appear elsewhere.

We use cohomology with coefficient group $\boldsymbol{Z}_2$. Ordinary cohomology is called $H^n(X)$. Cohomology with finite cochains is $H^n_f(X)$. By $\boldsymbol{Z}_2G$ we denote the group ring of G with coefficient ring $\boldsymbol{Z}_2$; modules, projective modules, etc., are with reference to this ring; if M is a module, $M^\star$ means $\mathrm{Hom}_{\boldsymbol{Z}_2G}(M, \boldsymbol{Z}_2G)$.

To say that a group G has infinitely many ends, means that $H^1(G; \boldsymbol{Z}_2G)$ is more than $\boldsymbol{Z}_2$. In terms of the regular covering space $\tilde{K}$, on which G acts freely with quotient complex K, where $H^1(\tilde{K})=0$, this means that $H^1_f(\tilde{K})$ contains more than two elements.

We suppose that K is a finite simplicial complex with ordered vertices; on this and on $\tilde{K}$ we have the standard cup-product of cochains defined, denoted by $\cdot$.

By a *minimal* 1-*cocycle* P we mean a finite 1-cocycle on $\tilde{K}$, which is nonzero in $H^1_f(\tilde{K})$, and which is, among all such, one involving the fewest 1-simplexes.

Since $H^1(\tilde{K})=0$, P cobounds two infinite 0-cochains E and E^*.

LEMMA 1. *E and E^* are connected.*

This means, in the case of E, say, that any two 0-simplexes in E can be joined by a finite chain of 1-simplexes, all of whose end-points lie in E. The reason for this is that otherwise P would be a disjoint sum of two cocycles, both non-trivial; one of these would be nonzero in $H_f^1(K)$ and smaller than P.

The group G acts on cocycles and cochains. If E and F are 0-cochains, then $E\cdot F$, the cup-product, is their intersection.

LEMMA 2. *If all these cochains are nonzero:*

$$E\cdot gE,\quad E\cdot gE^*,\quad E^*\cdot gE,\quad E^*\cdot gE^*$$

then P and gP have some 1-simplex in common.

For, we use Lemma 1 and the impossibility of writing P as a sum of disjoint cocycles to show that $P\cdot gP\neq 0$. If the simplicial cup product of two simplicial 1-cocycles is nonzero, they actually intersect.

LEMMA 3. *One, at least, of the 0-cochains in Lemma 2 is finite.*

For, one of them has coboundary involving fewer 1-simplexes than P.

LEMMA 4. *If g has infinite order, and if P and gP represent the same element in $H_f^1(\tilde{K})$, then $H_f^1(\tilde{K})=\mathbf{Z}_2$; i.e., G has two ends.*

Roughly speaking, $\tilde{K}$ is made up of a doubly infinite telescope whose sections are g^nF, where F is a finite 0-cochain with coboundary $P-gP$.

LEMMA 5. *If G has infinitely many ends and is torsion-free, and $1\neq g\in G$, then exactly one of the following is finite: $E\cdot gE$, $E\cdot gE^*$, $E^*\cdot gE$, $E^*\cdot gE^*$.*

For, by Lemma 3, at least one is finite. If two were finite, then P or $P-gP$ or gP would represent 0 in $H_f^1(\tilde{K})$, contradicting Lemma 4.

Thus, $G-\{1\}$ splits into four sets, denoted respectively

$$AA,\quad AA^*,\quad A^*A,\quad A^*A^*.$$

Formally, let X, Y, Z stand for A or A^*, and $(A^*)^*=A$.

LEMMA 6. (a) $(XY)^{-1}=YX$.
(b) $XY\cdot Y^*Z\subset XZ$.

(c) *For every* $g \in G$ *there is an upper bound to the numbers* n, *for which there are* $X_0, X_1, \cdots, X_n$, *and* $g_i \in X_{i-1}X_i^*$, *such that* $g = g_1 g_2 \cdots g_n$.
(d) *None of the sets* XY *is empty.*

The proof of this is mostly computational.

An *irreducible element* $g \in XY$ is one that cannot be written as $g_1 g_2$ for $g_1 \in XZ$, $g_2 \in Z^* Y$.

Let A denote $\{1\}$ together with all irreducible elements of AA. Let B denote $\{1\}$ together with all irreducible elements of A^*A^*. These are subgroups of G.

LEMMA 7. (a) *If there is no irreducible element of* AA^*, *then* G *is the free product of* A *and* B.

(b) *If there is an irreducible element* $\gamma \in AA^*$, *then* G *is the free product of* A *and the infinite cyclic group generated by* γ.

This is a combinatorial consequence of Lemma 6. It implies Theorem 1.

As for Theorem 2, it is not hard to prove that such a G can be represented as the group of covering translations of $\tilde{K}$ over a finite complex K, with $H^1(\tilde{K}) = 0$. Also, such a G is torsion-free.

LEMMA 8. *A torsion-free group with two ends is infinite cyclic.*

This is a theorem of C. T. C. Wall [4, Lemma 4.1]. Using this and Grushko's Theorem [2], we can prove Theorem 2 by induction on the number of generators of G, using Theorem 1 to split G into a free product, each factor of which has fewer generators. And so we need to prove:

LEMMA 9. *If* G *is a nontrivial, finitely generated group of cohomological dimension* 1, *then* $H^1(G; \boldsymbol{Z}_2G) \neq 0$.

Since G is 1-dimensional and finitely generated, the kernel M of the augmentation $\boldsymbol{Z}_2G \to \boldsymbol{Z}_2$ is a finitely generated projective module. The cohomology of G with coefficient group $\boldsymbol{Z}_2G$ now fits into an exact sequence:

$$0 \to H^0(G; \boldsymbol{Z}_2G) \to (\boldsymbol{Z}_2G)^\star \to (M)^\star \to H^1(G; \boldsymbol{Z}_2G) \to 0$$

Since G is finite dimensional, it has no elements of finite order. Hence G is infinite; this implies $H^0(G; \boldsymbol{Z}_2G) = 0$. If, additionally, $H^1(G; \boldsymbol{Z}_2G)$ were 0, the exact sequence would say that $(\boldsymbol{Z}_2G)^\star \to M^\star$ is an isomorphism. Since these are finitely generated projective modules, the original map $M \to \boldsymbol{Z}_2G$ would have to be an isomorphism, contrary to the fact that it has cokernel $\boldsymbol{Z}_2$.

This derivation of Theorem 2 from Theorem 1 was shown to us by C. T. C. Wall.

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ON THE NORM OF STABLE MEASURES[1]

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1. Limits of convolution powers and stable measures. Let $M(R)$ denote the Banach algebra of all complex-valued regular finite measures defined on the Borel sets of the real line R, where multiplication is defined by convolution, and

$$\|\mu\| = \sup \sum |\mu(R_i)|,$$

the supremum being taken over all finite collections of pairwise disjoint sets R_i whose union is R. Let $B(R)$ be the set of all Fourier transforms of measures in $M(R)$.

In [1], we characterized all possible limits

$$\lim_{n\to\infty} (\hat{\nu}(t/B_n))^n \exp(itA_n) = \hat{\mu}(t) \quad \text{for all } t \neq 0,$$

where $A_n \in R$, $B_n > 0$, $\hat{\nu}$, $\hat{\mu} \in B(R)$. This is a generalization of an old problem in probability theory (see e.g. [4]). One can show that a measure μ appears as a limit if and only if it is *stable*, i.e. has the following property: For all $a > 0$, $b > 0$ there exist $c > 0$ and $\gamma \in R$ such that

$$\hat{\mu}(at)\hat{\mu}(bt) = \hat{\mu}(ct) \exp(i\gamma t) \quad \text{for all } t \in R. \tag{1}$$

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