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GLOBAL ASYMPTOTIC ESTIMATES FOR ELLIPTIC SPECTRAL FUNCTIONS AND EIGENVALUES[1]

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The asymptotic behavior of the spectral function of a selfadjoint elliptic operator has been studied extensively; cf. the discussion in [1] and [7]. Recently Agmon and Kannai [2] and Hörmander [8] have obtained error estimates for general operators. Most of this work is concerned with interior estimates for operators with rather smooth coefficients. Here we consider behavior up to the boundary, with minimal assumptions on the coefficients. Details and proofs will appear elsewhere.

Let $\mathcal{C} = \sum a_\alpha(x)D^\alpha$ be an operator of order $m = 2r$ defined on a region Ω in R^n. We assume that the boundary $\partial\Omega$ is uniformly regular of class $m+1$ in the sense of [6]. Let $B_j = \sum b_{j,\beta}(x)D^\beta, j = 1, 2, \cdots, r$, be an operator of order $m_j < m$ defined on an ϵ-neighborhood of $\partial\Omega$. Suppose $0 < h \leq 1$. We assume

(1) $\mathcal{C}$ is uniformly strongly elliptic on Ω.

$(2)_h$ The coefficients a_α are bounded and measurable on Ω. For $|\alpha| = m$ and x, y in Ω,

$$| a_\alpha(y) - a_\alpha(x) | \leq c | y - x |^h.$$

$(3)_h$ The coefficients $b_{j,\beta}$ and their derivatives of order $\leq m - m_j$ are bounded and continuous on Ω. For $|\beta| = m_j$ and $|\gamma| = m - m_j$,

$$| D^\gamma b_{j,\beta}(y) - D^\gamma b_{j,\beta}(x) | \leq c | y - x |^h.$$

Let A be the operator in $L^2(\Omega)$ which is the restriction of $\mathcal{A}$ to the subspace

$$D(A) = \{u \in W^{m,2}(\Omega) \mid B_j u = 0 \quad \text{on} \quad \partial\Omega, \quad j = 1, \cdots, r\}.$$

Let $a(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$, $\xi \in R^n$, and set $a(x) = (2\pi)^{-n} \int_{a(x,\xi)<1} d\xi$.

THEOREM. *Suppose A is selfadjoint and semibounded. Let $A = \int \lambda dE_\lambda$ be the spectral resolution. Then each E_λ has a kernel $e_\lambda(x, y)$ which is continuous on the closure of $\Omega \times \Omega$. Moreover*

$$(1) \qquad\qquad | e_\lambda(x, y) | \leq c_0 \lambda^{n/m}.$$

For $x \in \Omega$, let $\delta(x)$ be the distance from x to $\partial\Omega$. Given $\epsilon > 0$,

$$(2) \quad \begin{aligned} | e_\lambda(x, x) &- a(x)\lambda^{n/m} | \\ &\leq c_1 \{\lambda^{n/m-\theta/m+\epsilon} + \lambda^{n/m} \exp(-c_2 \lambda^{1/2m}\delta(x)) + \delta(x)^{-2n}\}, \end{aligned}$$

where $\theta = h/(h+3)$.

For $x \in \partial\Omega$, there is a constant $a_1(x)$ such that given $\epsilon > 0$,

$$(3) \qquad\qquad | e_\lambda(x, x) - a_1(x)\lambda^{n/m} | \leq c_3 \lambda^{n/m-\theta/m+\epsilon}.$$

The constants c_j are independent of x, y.

One can give explicit conditions on the system $(\mathcal{A}, B_1, \cdots, B_r)$ which are necessary and sufficient for A to be selfadjoint and semi-bounded; cf. [3].

When Ω is bounded A has a complete set of orthonormal eigenfunctions $\{u_j\}$ corresponding to eigenvalues $\{\lambda_j\}$. Then

$$e_\lambda(x, y) = \sum_{\lambda_j \leq \lambda} u_j(x)\bar{u}_j(y).$$

In view of (1) and (2), integration of $e_\lambda(x, x)$ over Ω gives an estimate for the eigenvalues.

COROLLARY. *Suppose Ω is bounded, let $N(\lambda)$ be the number of eigenvalues of A which are $\leq \lambda$. For any $\epsilon > 0$,*

$$(4) \qquad N(\lambda) = \left(\int_\Omega a(x)dx\right)\lambda^{n/m} + O(\lambda^{n/m-\theta/m+\epsilon})$$

as $\lambda \to \infty$.

For $h = 1$ we have $\theta = 1/4$. The (interior) error estimate in [2], [8] has $\theta = 1/2$; however the coefficients are assumed to be C^∞ and essential use is made of this. In the C^∞ case our global estimate (2) can also be obtained with $\theta = 1/2$.

An analogue of the Theorem holds without the Hölder continuity assumptions, but with the weaker error estimate $o(\lambda^{n/m})$. This gives a correspondingly weaker version of (4), which can also be obtained by a quite different method [4], [5].

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