WIENER-HOPF OPERATORS AND ABSOLUTELY CONTINUOUS SPECTRA. II

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1. This paper is a continuation of [4]. It may be recalled that if A is a self-adjoint operator on a Hilbert space $\mathfrak{H}$ with spectral resolution $A=\int\lambda dE_\lambda$, then the set of elements x in $\mathfrak{H}$ for which $\|E_\lambda x\|^2$ is an absolutely continuous function of λ is a subspace, $\mathfrak{H}_a(A)$, of $\mathfrak{H}$ (see, e.g., Halmos [1, p. 104]). The operator A is said to be absolutely continuous if $\mathfrak{H}_a(A)=\mathfrak{H}$. As in [4], both spaces $L^2(0, \infty)$ and $L^2(-\infty, \infty)$ will be considered, but the underlying Hilbert space for the integral operators T and A occurring below will be $\mathfrak{H}=L^2(0, \infty)$.

As in [4], let $k(t)$ on $-\infty<t<\infty$ satisfy

$$(1) \qquad k \in L^1(-\infty, \infty) \cap L^2(-\infty, \infty) \quad \text{and} \quad k(-t) = \bar{k}(t),$$

and let $K(\lambda)$ denote the (real-valued) function

$$(2) \qquad K(\lambda) = \int_{-\infty}^{\infty} k(t)e^{i\lambda t}dt, \qquad -\infty < \lambda < \infty.$$

If the (bounded) operator T on $\mathfrak{H}$ is defined by

$$(3) \qquad (Tf)(t) = \int_0^t k(s-t)f(s)ds, \qquad 0 \leqq t < \infty,$$

then the self-adjoint operator $A=T+T^*=2\mathrm{Re}(T)$ is given by

$$(4) \qquad (Af)(t) = \int_0^\infty k(s-t)f(s)ds.$$

There will be proved the following

THEOREM. *If $k(t)$ satisfies* (1) *and if $k(t)\neq 0$ (a.e.) on $-\infty<t<\infty$, then the self-adjoint operator A of* (4) *is absolutely continuous and its spectrum is the closed interval*

$$(5) \qquad \mathrm{sp}(A) = [\inf K(\lambda), \sup K(\lambda)],$$

where $K(\lambda)$ is defined in (2).

In [4] the absolute continuity of A was established under the hypothesis that $K(\lambda)\neq 0$ a.e. According to the above Theorem how-

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ever, this property holds provided only that $K(\lambda)\not\equiv 0$, equivalently, that $k(t)\not\equiv 0$ a.e. In other words, A is absolutely continuous except in the trivial case $A=0$. (Similar assertions hold for self-adjoint Toeplitz operators; see [3, pp. 132–133] for a discussion and references.)[2] The relation (5) can be deduced from a theorem given in Krein [2, p. 224], concerning Wiener-Hopf operators on the half-line. However, this fact, as well as the assertions of the above Theorem concerning the absolute continuity, will be derived as consequences of general results on hyponormal operators on a Hilbert space and which are contained in the following

LEMMA. *Let T be a bounded operator on a Hilbert space $\mathfrak{H}$ and let*

$$T^*T - TT^* = C, \qquad C \geqq 0. \tag{6}$$

If $A=T+T^*$ then $\mathfrak{H}_a(A)$ contains the smallest subspace, $\mathfrak{M}_T$, of $\mathfrak{H}$ which reduces T and which contains the range of C. Moreover, the spectrum of the real part of T (that is, of $\frac{1}{2}A$) is the projection onto the real axis of the spectrum of T.

The proof of the Lemma can be found in [3, pp. 42–43, 46–47]. The proof of the absolute continuity assertion of the Theorem will be given in §2 and that of (5) in §3.

2. It will be convenient to recall a part of the argument given in [4]. It was noted there that relation (6) holds for T of (3) on $\mathfrak{H}=L^2(0,\ \infty)$ with $C=B^*B$ and $(Bf)(t)=\int_0^\infty k(t+s)f(s)ds$. For $f\in L^2(0,\ \infty)$, put $\hat{f}(\lambda)=\int_0^\infty e^{-i\lambda t}f(t)dt\equiv F_-(\lambda)$ and $F_+(\lambda)=\int_0^\infty e^{i\lambda t}f(t)dt$ and let R_+ and R_- denote the orthogonal subspaces of $L^2(-\infty,\ \infty)$ consisting of the elements F_+ and F_- respectively. (Note that R_+ [R_-] can be regarded as the space of Fourier transforms of elements in $L^2(-\infty,\ \infty)$ which are 0 on the right [left] half-line.)

As in [4], if the space $\mathfrak{M}_T$ of the Lemma is not $\mathfrak{H}$, then there exists a function $q\in\mathfrak{H}$, $q\not\equiv 0$ (that is, $q(t)\not\equiv 0$ a.e. on $0\leqq t<\infty$) such that $q\perp\mathfrak{M}_T$. (It will be shown below that necessarily $A=0$ in this case.) If $Q=Q(\lambda)=\int_0^\infty e^{-i\lambda t}q(t)dt$ $(\in R_-)$, then, as was shown in [4], $Q\perp\overline{K}_+^n R_+$, $n=0, 1, 2, \cdots$, where

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[2] *Added in proof.* It follows from a result of M. Rosenblum (*Self-adjoint Toeplitz operators,* 1965 Summer Institute in Spectral Theory and Statistical Mechanics, Brookhaven National Laboratory, Upton, New York) that the above A is unitarily equivalent to a self-adjoint Toeplitz matrix. The absolute continuity of A as well as the assertion (5) can then be deduced from the corresponding properties of Toeplitz operators; cf. [3] for a further discussion. This unitary equivalence is not used in the methods of the present paper however.

$$K_+(\lambda) = \int_0^\infty e^{i\lambda t} k(t)dt. \tag{8}$$

Since $Q \perp R_+$, it follows that

$$Q \perp (\mathrm{Re}(K_+))^n R_+ \quad \text{and} \quad Q \perp (\mathrm{Im}(K_+))^n R_+ \quad \text{for } n = 0, 1, 2, \cdots. \tag{9}$$

Only the first relation of (9) was exploited in [4], where it was shown that, as a consequence, $Q(\lambda) = 0$ a.e. on the set for which $\mathrm{Re}(K_+(\lambda)) \neq 0$. The same argument (involving Weierstrass' approximation theorem) shows however that $Q(\lambda) = 0$ a.e. also on the set for which $\mathrm{Im}(K_+(\lambda)) \neq 0$. Since $q(t) \not\equiv 0$ a.e. on $0 \leqq t < \infty$, then $Q(\lambda) \not\equiv 0$ a.e. on $-\infty < \lambda < \infty$, and hence $K_+(\lambda) = 0$ on a set of positive measure. However, since $k \in L^1(-\infty, \infty)$, $K_+(\lambda)$ is the boundary function of a function $K_+(z) = \int_0^\infty e^{izt} k(t)dt$ analytic in the upper half-plane $\mathrm{Im}(z) > 0$ and bounded and continuous on $\mathrm{Im}(z) \geqq 0$. If the half-plane $\mathrm{Im}(z) \geqq 0$ is mapped onto the unit circle $|w| \leqq 1$ by the linear fractional transformation $w = (z-i)/(z+i)$, one then obtains a function $K_+(z(w))$ analytic in $|w| < 1$, bounded on $|w| \leqq 1$, and continuous on $|w| \leqq 1$ except possibly at $w(\infty) = 1$, and which is 0 on the boundary $|w| = 1$ on a set of positive measure. It follows from the classical theorem of F. and M. Riesz [5] that this function must be identically 0 and hence, in particular, that $K_+(\lambda) \equiv 0$ on $-\infty < \lambda < \infty$. Hence $K(\lambda) = 2\mathrm{Re}(K_+(\lambda)) \equiv 0$ and so $k(t) \equiv 0$ a.e., that is, $A = 0$. This completes the proof of the first portion of the Theorem.

3. To prove (5), let z satisfy $0 < |z| < 1$ and put $\lambda = \int_0^\infty k(t) z^t dt$. (Here $z^t = e^{t \log z}$ where $\log z$ denotes any value of the logarithm function.) If $f(t) = z^t$, then $f \in \mathfrak{H}$ and one has

$$(T^* f)(t) = \int_t^\infty k(s-t) z^s ds = \int_0^\infty k(s) z^{t+s} ds = \lambda f(t). \tag{10}$$

Thus the range of $\int_0^\infty k(t) z^t dt$, for $0 < |z| < 1$, belongs to the spectrum, even the point spectrum, of T^*. (This fact and its derivation (10) are analogous to corresponding results for Toeplitz matrices, with the integrals replaced by power series, due to Wintner [6]; cf. also [3, p. 129].) Since the spectrum is a closed set, it follows that $K_+(\lambda)$ of (8) is in the spectrum of T^* for all real λ. Since $K(\lambda) = K_+(\lambda) + \overline{K}_+(\lambda)$, it follows from the second part of the Lemma that the interval $[\inf K(\lambda), \sup K(\lambda)]$ is certainly contained in the spectrum of $A = T + T^*$. (Only this much of the Lemma will be needed here.) On the other hand, as was shown in [4], $(Tf)^\wedge(\lambda) = \overline{K}_+(\lambda)\hat{f}(\lambda)$ and so, by the Parseval relation,

$$(Af,f) = (Tf,f) + (\overline{Tf,f}) = \int_{-\infty}^{\infty} K(\lambda)\,|\hat{f}(\lambda)|^2 d\lambda.$$

Hence, the spectrum of A is clearly a subset of $[\inf K(\lambda), \sup K(\lambda)]$ and the relation (5) is proved.

References

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