REPRESENTATION OF NONNEGATIVE CONTINUOUS FUNCTIONS ON PRODUCT SPACES

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The purpose of this note is to show that a nonnegative continuous function on a locally compact σ-compact product space is a countable sum of products of factor functions. That is, if f is a nonnegative continuous function on $\times_{\alpha \in A} X_\alpha$, there exist nonnegative continuous functions $g_{\alpha i}$, such that for each i, only finitely many $g_{\alpha i}$ are not identically 1, and for all x,

$$f(x) = \sum_i \prod_\alpha g_{\alpha i}(x_\alpha).$$

Since we are dealing with continuous real-valued functions, we may assume the spaces are completely regular. By local compactness and σ-compactness, we can reduce the problem to a countable number of representations on products of compact spaces. Each of these spaces can be represented as a closed subspace of a product of cubes. Then the function f^* defined on the product of the embeddings can be extended to the product of the cubes. Consequently, it is sufficient to prove the theorem when all X_α are $[0, 1]$.

Define the function

$$(1) \qquad \begin{aligned} h(x) &= 1 - |x|, \qquad & |x| &< 1, \\ &= 0, & |x| &\geqq 1. \end{aligned}$$

Then if $0 \leqq x \leqq 1$,

$$(2) \qquad 1 = \sum_{i=0}^{n} h(nx - i).$$

Let us proceed inductively as follows. Let $f_0 = f$, and when f_i is defined, $M_i = \max_x f_i(x)$. Then there exist $\alpha_1, \cdots, \alpha_q$, n such that

$$|f_i(x) - f_i(y)| < M_i/4 \text{ whenever } |x_{\alpha_t} - y_{\alpha_t}| < 1/n, \ t = 1, \cdots, q.$$

Now let

$$(3) \quad k_i(x) = \sum_{j_1=0}^{n} \cdots \sum_{j_q=0}^{n} \prod_{t=1}^{q} h(nx_{\alpha_t} - j_t) \max\left(0, f_i(u_j) - \tfrac{1}{4}M_i\right),$$

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where $u_{j_\alpha}=j_\alpha/n$, $\alpha=1, \cdots, q$, and u_{j_α} is arbitrary, say 0, otherwise.
We observe the following:

$$\sum_{j_i=0}^{n} \cdots \sum_{j_q=0}^{n} \prod_{t=1}^{q} h(nx_{\alpha_t} - j_t) = 1. \tag{4}$$

(5) All terms in (4) are nonnegative.

(6) If the jth term in (4) is positive, its coefficient in (3) lies between $f(x)-\frac{1}{2}M_i$ and $f(x)$.

Thus

$$f_{i+1}(x) = f_i(x) - k_i(x) \leqq \tfrac{1}{2}M_i. \tag{7}$$

Consequently, we may continue the process inductively and

$$f(x) = \sum_{i=0}^{\infty} k_i(x), \tag{8}$$

which is of the desired form.

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