

RESTRICTED LIE ALGEBRAS OF BOUNDED TYPE

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Introduction. It is known [13] that a Lie algebra over a modular field has indecomposable representations of arbitrarily high dimensionalities. If, however, the Lie algebra and its representations are required to be restricted (see [6, Chapter 5] for definitions), this need no longer be the case.

A restricted Lie algebra for which the degrees of its (restricted) indecomposable representations are bounded by some constant is said to be of *bounded type*; one for which this is not the case is said to be of *unbounded type*.

1. The simple three-dimensional Lie algebra, A_1 . Let A_1 be the split simple three-dimensional Lie algebra over the field K of characteristic $p > 3$. Then A_1 has a basis e, f, h with $[e, f] = h$, $[e, h] = 2e$, $[f, h] = -2f$ and with p -power mapping given by $e^p = f^p = 0$, $h^p = h$. There are p inequivalent irreducible (restricted) modules for A_1 , classified by their highest weight. Let M_λ , $0 \leq \lambda \leq p-1$, be the irreducible A_1 -module with highest weight λ , so that $[M_\lambda: K] = \lambda + 1$ [5].

Let U be the u -algebra [6] of A_1 and $U = \sum_{j=0}^{p-1} U_j$ its decomposition into its principal indecomposable modules (p.i.m.). Since U is a symmetric algebra [9] each U_j has a unique top and bottom composition factor, these are isomorphic, and each M_λ is isomorphic to the top composition factor of some U_j [2].

If M is an A_1 -module, denote by $M \sim M_{\lambda_1}, M_{\lambda_2}, \dots, M_{\lambda_r}$ the fact that the M_{λ_j} , in the given order, are the composition factors of some composition series for M .

THEOREM 1. Let $U(\lambda)$, $0 \leq \lambda \leq p-1$, be a p.i.m. of U whose top composition factor is isomorphic to M_λ . Then

- (i) $U(p-1) \cong M_{p-1}$ and $[U(p-1): K] = p$.
- (ii) If $\lambda \neq p-1$, then $U(\lambda) \sim M_\lambda, M_\gamma, M_\gamma, M_\lambda$, where $\lambda + \gamma = p-2$, and $[U(\lambda): K] = 2p$.

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The radical R of U is the two-sided ideal generated by the elements $e^{p-1}(h+1)$ and $(h+1)f^{p-1}$ [10]. It is not hard to show that R^2 is generated, as a two-sided ideal, by $e^{p-1}(h+1)f^{p-1}$, and $R^3=0$.

THEOREM 2. *Let M be an A_1 -module for which $MR^2 \neq 0$. Then M has a direct summand isomorphic to $U(\lambda)$ for some $\lambda \neq p-1$.*

THEOREM 3. *Let M be an indecomposable A_1 -module for which $M \supset MR \supset MR^2 = 0$. Then there exists a fixed λ , $0 \leq \lambda \leq p-2$, such that the socle of $M = MR =$ the direct sum of copies of M_λ , and M/MR is the direct sum of copies of M_γ , where $\lambda + \gamma = p-2$.*

COROLLARY. *If M is an indecomposable A_1 -module with exactly two composition factors, then $M \sim M_\lambda, M_\gamma$ for some $0 \leq \lambda \leq p-2$ and $\lambda + \gamma = p-2$, so that $[M: K] = p$.*

THEOREM 4. *For every positive integer k , there exists an indecomposable A_1 -module M with $[M: K] = k$, so that A_1 is of unbounded type.*

2. The classical Lie algebras [8]. Let T be a finite-dimensional associative algebra (with 1), and let P be a subalgebra (with 1) such that T is a finitely generated left and right P -module. If M is a right P -module, form $M \otimes_P T$ and define $(\sum_i m_i \otimes t_i)t = \sum_i m_i \otimes t_i t$ for $m_i \in M$ and $t, t_i \in T$. With this, $M \otimes_P T$ becomes a T -module, denoted by M^T . Suppose that for any indecomposable P -module M , the mapping $m \rightarrow m \otimes 1, m \in M$, is a P -isomorphism of M onto a P -direct summand of M^T . Then if P has indecomposable representations of arbitrarily high dimensionalities, the same is true of T (see [2], for example).

This is used to prove

THEOREM 5. *Let L be a restricted Lie algebra and A a restricted subalgebra of L such that $L = A \oplus \sum_{j=1}^n B_j$ (as vector spaces), where the B_j are restricted subalgebras with $[A, B_j] \subseteq B_j$. Then if A is of unbounded type, so is L .*

In particular, if L is a classical Lie algebra one may imbed the algebra A_1 in L in such a way that the conditions of Theorem 5 are satisfied.

THEOREM 6. *Let L be a classical Lie algebra over a field of characteristic $p > 3$. Then L is of unbounded type.*

Suppose L is a Lie algebra over the field F of characteristic $p > 3$, and that for some extension field K of F , the Lie algebra $L_K = L \otimes_F K$ is classical. L is called an F -form of the classical Lie algebra L_K .

It is known that K may be assumed to be a finite Galois extension of F [12, Chapter 4]. It follows from the next, more general, theorem that the forms of the classical Lie algebras are all of unbounded type.

THEOREM 7. *Let T be a finite-dimensional associative algebra (with 1) over the modular field F . Let K be a finite Galois extension of F such that $T_K = T \otimes_F K$ is of unbounded type. Then T is of unbounded type (over F).*

3. The Jacobson-Witt algebras and simple subalgebras.

THEOREM 8. *Let W_1 be the Witt algebra [4]. For $k = 1, 2, \dots$, there exists an indecomposable (restricted) representation of W_1 of dimension kp ($p > 2$ is the characteristic of the base field), so that W_1 is of unbounded type.*

Methods analogous to those used in the proof of Theorem 6 (where the part played by A_1 in that theorem is assumed here by W_1) may be used to prove

THEOREM 9. *The Jacobson-Witt algebras W_n [4] and their simple subalgebras $V_{m,\mu}$ [1] are all of unbounded type.*

It seems quite likely that all known simple restricted Lie algebras, i.e., those of classical and Cartan type [7], are of unbounded type.²

4. Nilpotent Lie algebras. Let L be a finite-dimensional restricted Lie algebra over the field K of prime characteristic p , and set $L^{(1)} = L$, $L^{(j+1)} = [L, L^{(j)}]$ for $j \geq 1$. If S is a subset of L , denote by S^p the linear span of the elements s^p , $s \in S$, so that $S^{p^{k+1}} \subseteq (S^{p^k})^p$.

DEFINITION. Set $L_1 = L$ and $L_k = L^{(k)} + (L^{(k-1)})^p + (L^{(k-2)})^{p^2} + \dots + L^{p^{k-1}}$ for $k > 1$. The series $L = L_1 \supseteq L_2 \supseteq L_3 \supseteq \dots$ is the (restricted) lower central series for L , and if $L_k = 0$ for some $k > 1$, L is called *nil*; if $L^{(k)} = 0$ for some $k > 1$, L is called *nilpotent*.

The L_k are ideals in L , $L_k = L^{(k)} + (L_{k-1})^p$, and if L is nil, then it is also nilpotent (although the converse is not true) and $L^{p^s} = 0$ for s sufficiently large [3]. An element $x \in L$ is nilpotent of index k if $x^{p^{k-1}} \neq 0$ and $x^{p^k} = 0$ for some $k > 0$; the subalgebra generated by x , i.e. the linear span of $x, x^p, x^{p^2}, \dots, x^{p^{k-1}}$, is called nil-cyclic and is denoted by $\langle x \rangle$. An element $h \in L$ is called separable if $h \in \langle h^p \rangle$, the subalgebra generated by h^p , and a commutative (restricted) Lie algebra consisting of separable elements is called toral.

The u -algebra of a nil-cyclic Lie algebra is the group algebra of a

² Robert Wilson has shown that these same methods can be used to prove that the remaining algebras of Cartan type are also of unbounded type.

cyclic p -group and hence is of bounded type. On the other hand, the u -algebra of the direct sum of $n > 1$ nil-cyclic Lie algebras is the group algebra of the direct product of $n > 1$ cyclic p -groups, and so is of unbounded type. This is used to prove

LEMMA A. *Let L be a nil Lie algebra over the modular field K . Then L is of bounded type if and only if L is nil-cyclic.*

LEMMA B. *Let $L = H \oplus \langle x \rangle$ (semidirect) over the algebraically closed field K of characteristic $p > 0$, where H is a toral subalgebra and $\langle x \rangle$ is the nil-cyclic ideal generated by the nilpotent element x of index $k > 0$. Then every indecomposable (restricted) L -module has dimension $\leq p^k$ and is a homomorphic image of a p .i.m. of the u -algebra of L .*

THEOREM 10. *Let L be a nilpotent (restricted) Lie algebra over an algebraically closed field of characteristic p . Then L is of bounded type if and only if $L = H \oplus \langle x \rangle$, where H is a toral ideal and $\langle x \rangle$ is the nil-cyclic ideal generated by the nilpotent element x (either H or x may be trivial).*

PROOF. Suppose L is of bounded type; we may assume L is not nil. One may take $H = L_k$, where $L_k = L_{k+1}$ and $L_k = (L_k)^p$. If L/H is to be of bounded type, it is nil-cyclic and one then shows that the extension $0 \rightarrow H \rightarrow L \rightarrow L/H \rightarrow 0$ splits. The converse is clear.

5. Solvable Lie algebras. Even over an algebraically closed field, a solvable restricted Lie algebra need not satisfy Lie's theorem, that is, a representation of a solvable Lie algebra need not be simultaneously triangularizable. If, however, $L = H \oplus N$ (semidirect), where H is a toral subalgebra and N is a nil ideal, L is said to be *strongly solvable* and does satisfy Lie's theorem. In fact, this property characterizes strongly solvable Lie algebras. Over a perfect field, a solvable Lie algebra L is strongly solvable if and only if $[L, L]$ consists of nilpotent elements, in which case the set of all nilpotent elements of L forms a maximal nil ideal [11].

LEMMA C. *Let $L = H \oplus N$ (semidirect) over an algebraically closed field of characteristic $p \geq 3$, where H is a toral subalgebra and N is a nil ideal with basis e, f , where $[e, f] = 0$ and $e^p = f^p = 0$. Then L is of unbounded type.*

THEOREM 11. *A strongly solvable restricted Lie algebra over an algebraically closed field of characteristic $p \geq 3$ is of bounded type if and only if its maximal nil ideal is nil-cyclic.*

PROOF. Let $L = H \oplus N$, H a toral subalgebra, and N the maximal

nil ideal of L ; assume neither H nor N is trivial and that N is not nil-cyclic. If $N' = [N, N] + N^p$, then N' is an ideal of L and $\bar{N} = N/N'$ is a commutative Lie algebra with trivial p -power mapping. Then $L/N' = \bar{H} \oplus \bar{N}$ (semidirect), where $\bar{H} \cong H$ and \bar{N} is the direct sum of one-dimensional \bar{H} -modules. The fact that N is nil but not nil-cyclic implies $\dim \bar{N} \geq 2$, and we may, if necessary, factor N/N' by an ideal of codimension 2 and so assume $\dim \bar{N}$ is exactly 2. From Lemma C it follows that L/N' , hence L , is of unbounded type. The converse is clear.

A Lie algebra L with center C is said to be *centrally strongly solvable* if L/C is strongly solvable; it is not hard to see that these algebras are characterized by the nilpotency of $[L, L]$.

THEOREM 12. *A centrally strongly solvable restricted Lie algebra L over the algebraically closed field K of characteristic $p \geq 3$ is of bounded type if and only if either*

(i) L is strongly solvable with nil-cyclic maximal nil ideal or (ii) $L = H \oplus Ke$ (as vector spaces), where H is a toral subalgebra and $0 \neq [e, H] \subseteq Ke$, $0 \neq e^p \in H$.

DEFINITION. Let L be a restricted Lie algebra and M a finite-dimensional L -module. M may be considered as a commutative restricted Lie algebra with trivial p -power mapping. The *split extension* of L by M is the restricted Lie algebra L^M whose underlying space is $L \oplus M$, with product and p -power mapping determined by the action of L on M (see [6]).

It is clear that M is a nil ideal of L^M and that if L^M is of bounded type, so is L . Partial conditions for a converse are included in the following:

THEOREM 13. *Let L be a restricted Lie algebra over the algebraically closed field K of characteristic $p > 3$ such that L is of bounded type and either*

- (i) toral,
- (ii) strongly solvable, or
- (iii) centrally strongly solvable.

Let $M \neq 0$ be a finite-dimensional L -module. Then the split extension L^M is of bounded type if and only if L is toral and M is 1-dimensional.

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