GEOMETRIC PROGRAMMING: A UNIFIED DUALITY THEORY FOR QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMS AND l_p-CONSTRAINED l_p-APPROXIMATION PROBLEMS[1]

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The duality theory of geometric programming as developed by Duffin, Peterson, and Zener [1] is based on abstract properties shared by certain classical inequalities, such as Cauchy's arithmetic-geometric mean inequality and Hölder's inequality. Inequalities with these abstract properties have been termed "geometric inequalities" ([1, p. 195]). We have found a new geometric inequality, which we state below, and have used it to extend the "refined duality theory" of geometric programming developed by Duffin and Peterson ([2] and [1, Chapter VI]). This extended duality theory treats both quadratically-constrained quadratic programs and l_p-constrained l_p-approximation problems. By a quadratically constrained quadratic program we mean: to minimize a positive semidefinite quadratic function, subject to inequality constraints expressed in terms of the same type of functions. By an l_p-constrained l_p-approximation problem we mean: to minimize the l_p norm of the difference between a fixed vector and a variable linear combination of other fixed vectors, subject to inequality constraints expressed by means of l_p norms.

Both the classical unsymmetrical duality theorems for linear programming (Gale, Kuhn and Tucker [3], and Dantzig and Orden [4]) and the unsymmetrical duality theorems for linearly-constrained quadratic programs (Dennis [5], Dorn [6], [7], Wolfe [8], Hanson [9], Mangasarian [10], Huard [11], and Cottle [12]) can be derived from the extended duality theorems that we state below and have proved on the basis of the new geometric inequality.

The new geometric inequality is

$$\sum_{1}^{N+1} x_i y_i \leqq y_{N+1}\left(\sum_{1}^{N} p_i^{-1} \left| x_i - b_i \right|^{p_i} + (x_{N+1} - b_{N+1}) \right) + \sum_{1}^{N} \left(q_i^{-1} y_{N+1}^{(1-q_i)} \left| y_i \right|^{q_i} + b_i y_i \right) + b_{N+1} y_{N+1},$$

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which is valid for each $\mathbf{x}$ in E_{N+1} and each $\boldsymbol{y}$ in the cone

$$T = \{\boldsymbol{y} \in E_{N+1} \mid y_{N+1} \geqq 0, \text{ and } y_{N+1} = 0 \text{ only if } \boldsymbol{y} = \mathbf{0}\},$$

with the understanding that $\sum_1^N q_i^{-1} y_{N+1}^{(1-q_i)} |y_i|^{q_i}$ is defined to be zero when $\boldsymbol{y}=\mathbf{0}$. Here $\boldsymbol{b}=(b_1, b_2, \cdots, b_{N+1})$ is an arbitrary, but fixed, vector in E_{N+1}, and p_i and q_i are arbitrary, but fixed, real numbers that satisfy the conditions $p_i, q_i > 1$ and $1/p_i + 1/q_i = 1$, $i=1, 2, \cdots, N$.

Every quadratically-constrained quadratic program and every l_p-constrained l_p-approximation problem are special cases of the following program.

PRIMAL PROGRAM A. *Find the infimum of $G_0(\mathrm{x})$ subject to the following constraints on* x.

(1) $G_k(\mathrm{x}) \leqq 0$ *for each k in* $\{1, 2, \cdots, r\}$.

(2) $\mathrm{x} \in \mathcal{P}$.

Here

$$G_k(\mathrm{x}) = \sum_{[k]} p_i^{-1} |x_i - b_i|^{p_i} + (x_{]k[} - b_{]k[}), \quad k = 0, 1, 2, \cdots, r,$$

where

$$[k] = \{m_k, m_k + 1, \cdots, n_k\}, \quad k = 0, 1, 2, \cdots, r,$$

$$]k[= n_k + 1, \qquad k = 0, 1, 2, \cdots, r,$$

and

$$m_0 = 1, m_1 = n_0 + 2, m_2 = n_1 + 2, \cdots, m_r = n_{r-1} + 2, n_r + 1 = n.$$

The vector $\boldsymbol{b}=(b_1, b_2, \cdots, b_n)$ is an arbitrary, but fixed, vector in E_n, and the arbitrary, but fixed, constants p_i satisfy the condition $p_i > 1$ for each i in $[k]$, $k=0, 1, 2, \cdots, r$. The set $\mathcal{P}$ is a fixed, but arbitrary, vector subspace of E_n.

To put an arbitrary quadratically-constrained quadratic program with m independent variables $z_1, \cdots, z_m$ into the form of primal program A, first observe that each positive semidefinite quadratic function $(\frac{1}{2})z^tC_z + c^tz$ can be factored as $(\frac{1}{2})(Dz)^t(Dz) + \mathbf{c}^tz$ where D is an appropriate $m \times m$ matrix. In particular, the objective function for such a program can be factored to give a matrix D_0 and a row vector $\mathbf{c}_0{}^t$. Correspondingly, the kth constraint can be factored to give a matrix D_k and a row vector $\mathbf{c}_k{}^t$. Let $M = [D_0, \mathbf{c}_0{}^t, D_1, \mathbf{c}_1{}^t, \cdots, D_r, \mathbf{c}_r{}^t]^t$ and specialize primal program A by letting

$$n_k = (k+1)m + k, \qquad k = 0, 1, \cdots, r$$

$$p_i = 2 \text{ for each } i \in [k], \qquad k = 0, 1, \cdots, r,$$

$$b_i = 0 \text{ for each } i \in [k], \qquad k = 0, 1, \cdots, r.$$

Finally, identify $\mathcal{P}$ with the column space of the above matrix M; that is, let $\mathbf{x}=M\mathbf{z}$. With these specializations, primal program A is equivalent to the quadratically-constrained quadratic program.

All linearly-constrained quadratic programs can be obtained from the most general quadratically-constrained quadratic program by choosing the ith row vector of M equal to $\mathbf{0}$ for each i in $[k]$, $k=1, 2, \cdots, r$. Moreover, all linear programs can be obtained by further restricting M so that its ith row vector equals $\mathbf{0}$ for each i in $[0]$.

To obtain from primal program A the most general l_p-constrained l_p-approximation problem with m spanning vectors, choose $p_i=p_k$ for each i in $[k]$, $k=0, 1, \cdots, r$, and identify $\mathcal{P}$ with the column space of an arbitrary $n\times m$ matrix M with $]k[$th row vector equal to $\mathbf{0}$ for $k=0, 1, 2, \cdots, r$.

The dual program corresponding to primal program A is

DUAL PROGRAM B. *Find the supremum of* $v(\mathbf{y})$ *subject to the following constraints on* $\mathbf{y}$.

(1) $y_{]0[}=1$.

(2) *For each integer* k *in* $\{1, 2, \cdots, r\}$ *the vector component* $y_{]k[}\geqq 0$, *and* $y_{]k[}=0$ *only if* $y_i=0$ *for each* i *in* $[k]$.

(3) $\mathbf{y}\in\mathcal{D}$.

Here

$$v(\mathbf{y}) = -\sum_{k=0}^{r}\left\{\sum_{[k]}\left(q_i^{-1} y_{]k[}^{(1-q_i)}\,|y_i|^{q_i} + b_i y_i\right) + b_{]k[}y_{]k[}\right\}$$

where $[k]$, $]k[$, *and* r *are as defined in primal program* A. *The fixed vector* $\mathbf{b}=(b_1, b_2, \cdots, b_n)$ *is identical to the vector* $\mathbf{b}$ *of primal program* A, *and the constants* q_i *are determined from the constants* p_i *of primal program* A *by the condition*

$$1/p_i + 1/q_i = 1 \qquad \textit{for each } i \textit{ in } [k], \quad k = 0, 1, 2, \cdots, r.$$

The subset $\mathcal{D}$ *of* E_n *is the orthogonal complement of the vector subspace* $\mathcal{P}$ *of primal program* A.

If for primal program A we take any linearly-constrained quadratic program, or any linear program, then correspondingly dual program B reduces to the well-known dual program. To recognize this, two recollections and two elementary observations are needed. First, recall that appropriate row vectors of the matrix M must be set equal to $\mathbf{0}$ and then observe that the dual variable y_i corresponding to such a row vector is essentially unconstrained. Second, recall that $b_i=0$

for each i in $[k]$, $k=0, 1, 2, \cdots, r$, and then observe from the form of the resulting dual objective function v that when the variable y_i is essentially unconstrained for some i in some $[k]$ this variable y_i can be set equal to zero without changing the constrained supremum of $v(y)$. In the linearly-constrained quadratic case v reduces to the quadratic function

$$v(y) = -\frac{1}{2}\sum_{[0]} y_i^2 - b_{]0[} - \sum_{1}^{r} b_{]k[}y_{]k[}.$$

In the completely linear case v further reduces to the linear function

$$v(y) = -b_{]0[} - \sum_{1}^{r} b_{]k[}y_{]k[}.$$

We shall use the following standard terminology in stating our duality theorems. The ***objective function*** for a program is the function to be optimized (G_0 or v in our theory). A program is ***consistent*** if there is at least one point that satisfies its constraints. Each such point is said to be a ***feasible solution*** to the program. The constrained infimum (supremum) of the objective function for a consistent program is termed the ***infimum (supremum) of the program.*** A feasible solution to a program is ***optimal*** if the resulting value of the objective function is actually equal to the infimum (supremum) of the program. The infimum (supremum) of a program with an optimal feasible solution is said to be the ***minimum (maximum) of the program.*** Thus a program can have an infimum (supremum) without having a minimum (maximum), but not conversely.

The following existence theorem relates primal program A and its dual program B.

THEOREM 1. *If primal program* A *is consistent, then it has a finite infimum* M_A *if, and only if, its dual program* B *is consistent. If dual program* B *is consistent, then it has a finite supremum* M_B *if, and only if, its primal program* A *is consistent.*

Unlike Theorem 1, the following duality theorem is not symmetrical relative to primal program A and its dual program B.

THEOREM 2. *If primal program* A *and its dual program* B *are both consistent, then*

(I) *Program* A *has a finite infimum* M_A *and program* B *has a finite supremum* M_B, *with* $M_A = M_B$.

(II) *The infimum* M_A *of program* A *is actually a minimum.*

(III) *The supremum M_B of program* B *is actually a maximum if, and only if, there are nonnegative Kuhn-Tucker (Lagrange) multipliers that solve the saddle-point problem for program* A.

Conclusion III can be strengthened so as to make Theorem 2 symmetrical relative to primal program A and its dual program B, if the class of programs is restricted to those programs for which primal program A has linear constraints. This strengthening is due to the well-known fact that each convex program with linear constraints has Kuhn-Tucker multipliers if it has a minimum.

The following duality theorem characterizes the sets of optimal feasible solutions.

THEOREM 3. *Given an optimal feasible solution* $\mathbf{x}'$ *to primal program* A, *a feasible solution* $\mathbf{y}$ *to dual program* B *is optimal if, and only if,* $\mathbf{y}$ *satisfies the conditions*

$$y_{]k[}G_k(\mathbf{x}') = 0, \quad k = 1, 2, \cdots, r,$$

and

$$y_i = y_{]k[}(\operatorname{sgn}[x_i' - b_i])\,|x_i' - b_i|^{p_i-1}, \quad i \in [k], \ k = 0, 1, \cdots, r.$$

For such an optimal feasible solution $\mathbf{y}'$ *the numbers* $y'_{]k[}$, $k = 1, 2, \cdots, r$, *are Kuhn-Tucker multipliers for primal program* A. *Given an optimal feasible solution* $\mathbf{y}'$ *to dual program* B, *a feasible solution* $\mathbf{x}$ *to primal program* A *is optimal if, and only if,* $\mathbf{x}$ *satisfies the conditions*

$$y'_{]k[}G_k(\mathbf{x}) = 0, \qquad k = 1, 2, \cdots, r,$$

and

$$x_i = (\operatorname{sgn} y_i')(|y_i'|/y'_{]k[})^{(q_i-1)} + b_i, \quad i \in [k], \quad k \in P,$$

where

$$P = \{k \mid y'_{]k[} > 0\}.$$

Proofs for the preceding theorems will be given elsewhere. Sensitivity analyses and a computational algorithm that takes advantage of the essentially linear of the dual constraints will be described in forthcoming papers.

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