k-MERSIONS OF MANIFOLDS[1]

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Let M^n be an n-dimensional C^∞ manifold and W^p be a p-dimensional C^∞ manifold. A C^∞ mapping $f: M^n \to W^p$ is called a k-mersion if its rank is greater than or equal to k everywhere. The set of k-mersions, endowed with the C^1 topology, is denoted $R(M^n, W^p; k)$. A k-regular homotopy between k-mersions f and g is a continuous mapping $F: I \to R(M^n, W^p; k)$ such that $F(0) = f$ and $f(1) = g$.

A k-bundle map, $\psi: TM^n \to TW^p$ between the tangent spaces of M^n and W^p is a continuous fibre preserving mapping such that the restriction of ψ to any fibre is a linear map of rank at least k. The space of k-bundle maps with the compact open topology is denoted $T(M^n, W^p; k)$.

An n-mersion is an immersion, and an n-regular homotopy is usually called a regular homotopy. In 1958 and 1959, Smale [4], [5] published papers classifying immersions of spheres in Euclidean spaces. Smale proved that if $n < p$, the regular homotopy classes of immersions of S^n in E^p are in one to one correspondence with the homotopy classes of sections of S^n into the bundle associated with TS^n whose fibre is the Stiefel manifold $V_{p,n}$ of n frames in p-dimensional Euclidean space. Smale obtained this classification by proving a stronger result, namely, that the map $d: R(S^n, E^p; n) \to T(S^n, E^p; n)$ defined by $d(f) = df$ is a weak homotopy equivalence if $n < p$. His proof was based on the diagram

$$\begin{array}{ccc} R(S^n, E^p; n) & \xrightarrow{d} & T(S^n, E^p; n) \\ \downarrow i^* & & \downarrow j^* \\ R(D^n, E^p; n) & \xrightarrow{d} & T(D^n, E^p; n) \end{array} \tag{1}$$

where D^n is identified with a hemisphere of S^n, and i^* and j^* are restriction maps. The main step in the proof consists of showing that i^*

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and j^* are fiberings (i.e., have covering homotopy property). Then it is easily shown that the d of the bottom row is a weak homotopy equivalence and that d restricted to a fibre of i^* is a weak homotopy equivalence. It follows immediately from the homotopy sequence of a bundle and the five lemma that the d of the top row is a weak homotopy equivalence.

In 1959, Hirsch [1] extended this result to the case of immersions $R(M^n, W^p; n)$ of a C^∞ manifold in another, with dim $M^n <$ dim W^p, (i.e. $n<p$) and $\partial W^p = \emptyset$. Poenaru's exposition of this result [3] was the basis of Phillips' thesis in 1965 (published as [2]) which stated that if M^n has no compact components with empty boundary and $\partial W^p = \emptyset$, then $d\colon R(M^n, W^p; p) \to T(M^n, W^p; p)$ is a weak homotopy equivalence. Phillips called the maps whose rank equalled the dimension of the image space "submersions."

Poenaru's exposition also is the basis of the generalization given here.

THEOREM 1. *Let M^n and W^p be C^∞ manifolds with $\partial W^p = \emptyset$. The mapping $d\colon R(M^n, W^p; k) \to T(M^n, W^p; k)$ defined by $d(f) = df$ is a weak homotopy equivalence if either*

(a) *M^n has no compact components with empty boundary, or*

(b) *$k < p$.*

COROLLARY 1. *If condition* (a) *or condition* (b) *of Theorem* 1 *is satisfied, the k-regular homotopy classes of k-mersions of M^n in W^p are in one to one correspondence with the homotopy classes of k-bundle maps of TM^n in TW^p.*

Denote by $M^*(p, n; k)$ the set of $p \times n$ matrices of rank at least k.

COROLLARY 2. *Suppose condition* (a) *or condition* (b) *is satisfied. The k-regular homotopy classes of k-mersions of M^n in R^p are in one to one correspondence with the homotopy classes of sections of M^n into the bundle associated with TM^n whose fibre is $M^*(p, n; k)$.*

As an application of the above, it can be shown that if $p \geqq (3/2)(n-1)$, there exists an $n-1$ mersion of M^n in R^p.

The proof of Theorem 1 uses a filtration of M^n

$$D^n = U_0^n \subset U_1^n \subset U_2^n \cdots \subset M^n$$

where U_i is obtained from U_{i-1} by adding a handle, essentially. The simple scheme of (1) is replaced by

$$\begin{array}{ccc}
R(M^n, W^p; k) & \xrightarrow{d} & T(M^n, W^p; k) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
R(U_2^n, W^p; k) & \xrightarrow{d} & T(U_2, W^p; k) \\
\downarrow i_2^* & & \downarrow j_2^* \\
R(U_1^n, W^p; k) & \xrightarrow{d} & T(U_1^n, W^p; k) \\
\downarrow i_1^* & & \downarrow j_1^* \\
R(D^n, W^p; k) & \xrightarrow{d} & T(D^n, W^p; k)
\end{array}$$

It is easy to show that the d of the bottom row is a weak homotopy equivalence, and that all of the j^* maps are fiberings. The main step in the proof is in showing that the i^* maps are fiberings from which it also easily follows that d restricted to a fibre is a weak homotopy equivalence. The rough outlines of the proof that the i^* maps are fiberings are as follows. Let V^n be obtained from U^n by adding a handle of index λ. Denote $I^m = ([0, 1])^m$, and $I^{m-1}([0, 1])^{m-1} \times \{0\}$. The map i^* has the covering homotopy property if, given continuous maps $g\colon I^m \to R(U^n, W^p; k)$ and $G\colon I^{m-1} \to R(V^n, W^p; k)$ such that $G(q) | U^n = g(q)$ when $q \in I^{m-1}$, there is a mapping $\overline{G}\colon I^m \to R(V^n, W^p; k)$ with

$$(3) \qquad \overline{G}(q) = G(q), \quad q \in I^{m-1}, \qquad \overline{G}(q) \,|\, U^n = g(q), \quad q \in I^m.$$

$\overline{G}(q)$ is an extension of $g(q)$ over V, so $\overline{G}$ can be viewed as a continuously varying set of extensions of g over V which agrees with G on I^{m-1}.

If, for each $f \in R(U^n, W^p; k)$ there is a neighborhood $\eta(f)$ for which $i^*_{\eta(f)} = (i^* | i^{*-1}(\eta(f)))$ has the covering homotopy property, it is very easy to show that i^* has the covering homotopy property. Thus, given $f \in R(U^n, W^p; k)$ it suffices to find a neighborhood η of f such that if g and G map into η and $i^{*-1}(\eta)$ respectively, an extension $\overline{G}$ satisfying (3) can always be constructed.

As a first step, it is not difficult to show that there is a neighborhood η of f whose elements can be factored through automorphisms of some compact $(n+p)$-dimensional manifold C. In fact, there is a compact neighborhood N^n of U^n in V^n, an embedding $s\colon N^n \to C$, a differentiable map $P\colon C \to W^p$, and a continuous mapping $\nu\colon \eta \to \operatorname{Aut} C$ (the automorphisms of C which equal the identity near ∂C) so that $P \circ \nu(h) \circ s | U = h$, $h \in \eta$.

$$\begin{array}{ccccc} & & C \xrightarrow{\ \nu(h)\ } C & & \\ (4) \qquad & s\,|\,U \nearrow & & \searrow P & \\ & U & \xrightarrow{\qquad\qquad h \qquad\qquad} & & W^p \end{array}$$

Note that $P \circ \nu(h) \circ s$ is an extension of h to N^n, and if $g: I^m \rightarrow \eta$, then $P \circ \nu(g(q)) \circ s$ is a continuously varying set of extensions of g over N. It turns out that ν can be modified so that $P \circ \nu(g(q)) \circ s = G(q) | N$ if $q \in I^{m-1}$. Thus the factorization (4) enables a lift to be constructed over a part of the handle, at least.

Near ∂C, $\nu(h)$ is always the identity. The second part of the construction uses this fact. Let $\dot{N}$ be the boundary of N^n in V^n, i.e., $\dot{N} = (\partial N^n - \partial V^n)^-$. Since N^n is embedded in C, we can assume $N^n \subset C$. If N^n can be deformed in C through embeddings ξ_t, $t \in [0, 1]$, so that U^n is always left fixed, but $\dot{N}$ is carried out to ∂C (i.e., $\xi_1(\dot{N}) \subset \partial C$), then for x near $\dot{N}$, $P \circ \nu(g(q)) \circ \xi_1(x) = P \circ \xi_1(x)$ for all $q \in I^m$. Using this fact, it is not too difficult to piece together an explicit formula defining $\overline{G}$. The only use of the hypotheses, condition (a) or condition (b), is in proving the existence of such a deformation ξ_t. This turns out to be simple in case (a) but case (b) is quite complicated when the index of the handle is n.

The details of the proof and a number of applications will be presented in a forthcoming paper.

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