THE FOLDED RIBBON THEOREM FOR REGULAR CLOSED CURVES IN THE PLANE

BY GEORGE K. FRANCIS[1]

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Let S be the oriented circle with base point, E the oriented Euclidian plane, and V the positively oriented two frames in E. Let L be the space of C^1-regular immersions $g: S \to E$ with continuous right transverse field $\hat{g}$. For $g \in L$, set $\partial g = (g, \hat{g}, g'): S \to E \times V$. A positive monotone regular homotopy (=monotopy) from loop g_{-1} to g_{+1} is a C^1-regular homotopy $G: [-1, +1] \times S \to E$ with positive Jacobian and $\partial G(i, x) = \partial g_i(x)$, $i = \pm 1$, where $\partial G = (G, \partial G/\partial t, \partial G/\partial x)$. A negative monotopy G from g_{-1} to g_{+1} is such that $G^*(t, x) = G(-t, x)$ is a positive monotopy from g_{+1} to g_{-1}. A monotopy is stronger than a regular homotopy in that the latter requires only that $\partial G/\partial x \neq 0$. The tangent winding number TWN of g in L is the degree of $g'/|g'|: S \to S$. Because degree is a homotopy invariant, regular homotopy preserves the TWN. The converse of this is the Whitney-Graustein Theorem [3]. The TWN actually classifies L in a much stronger fashion.

THEOREM. *For two regular loops g_i, $i = \pm 1$, of like TWN, there always is a regular loop g_0 and two monotopies $H_i: g_i \sim g_0$, $i = \pm 1$, of like sign equal to sign ($TWN \pm \frac{1}{2}$).*

Note that TWN $= 0$ belongs to both cases. For TWN $= 1$, two concentric circles are monotopic. Not so for two circles whose interiors are disjoint; yet each is monotopic to a circle surrounding them both.

The method of proof is entirely constructive. The normal loops L_N have only simple, signed, transverse self-intersections (=nodes). L_N is dense and open (=generic) in L under the topology induced by $\|g-h\| = \max |\partial g(x) - \partial h(x)|$, $x \in S$. (See [3] for details.)

PROPOSITION 1. *If $g \in L$ and $\epsilon > 0$, there is an $h \in L_N$ with $\|g-h\| < \epsilon$ and a monotopy of prescribed sign between them.*

The proof of Proposition 1 makes use of a stable condition of "parallelity": $\min \det (g(x) - h(x), tg'(x) + (1-t)h'(x)) > 0$, over all $x \in S$ and $t \in [0, 1]$. The key lemma reads:

LEMMA. *If w is a continuous, periodic, transverse field along the ordinate in the (t, x)-Cartesian plane, then the map $F(t, x) = (t - z(t), x)$*

[1] This work comprised part of the author's dissertation written under Professor Charles J. Titus at the University of Michigan, 1967.

$+\frac{1}{2}\int_{x-z}^{x+z} w(s)ds$ *is a diffeomorphism of* $(0 \leqq t \leqq 1)$ *with positive Jacobian, for a suitable bump-function* $z(t)$.

Suppose G is a monotopy terminating with loop g. If d is a right-framed embedding of a parameter interval $[a, b]$ on which g is one-to-one, such that $d[a, b] \cup g[a, b]$ is a Jordan loop, with $\partial d(c) = \partial g(c)$, $c = a, b$, and $\operatorname{sgn}(d, g)$ is given by

$$\lim(x \to a+) \operatorname{sgn} \det(d'(x), g'(x)) = \lim(x \to b-) \operatorname{sgn} \det(g'(x), d'(x)),$$

it is called a simple detour of g with $\operatorname{supp}(d) = [a, b]$. It operates on g to give the loop $dg(x) = d(x)$ for $x \in \operatorname{supp}(d)$, $= g(x)$ otherwise.

PROPOSITION 2. *If the sign of* G *is the same as that of* d, *there is a modified monotopy* dG, *terminating in* dg.

Suppose d_1 is a simple detour of g, and d_2 is a simple detour of $d_1 g$. If $\operatorname{supp}(d_1) \cap \operatorname{supp}(d_2) = \emptyset$, d_1 is also a detour of $d_2 g$. Write $(d_1 + d_2)g = d_2(d_1 g)$ with $\operatorname{supp}(d_1 + d_2) = \operatorname{supp}(d_1) \cup \operatorname{supp}(d_2)$. If $\operatorname{supp}(d_i) \supset \operatorname{supp}(d_j)$, $i \neq j$, and both signs are the same, write $(d_2 d_1)g = d_2(d_1 g)$, where $\operatorname{supp}(d_2 d_1)$ is the larger of the two nested intervals. If d is a detour of g, set $d^* = g \mid \operatorname{supp}(d)$. Then d^* is a detour of dg, of sign opposite to that of d and with the same support. $(d^* d)g = g$ undetoured.

A formal expression D, made up of a finite number of legitimate sums and products of simple detours is a monotone compound detour provided all components have the same sign; otherwise it is a mixed compound.

PROPOSITION 3. *Suppose* $G_{-1}: [-1, 0] \times S \to E$ *is a monotopy* $g_{-1} \sim h_{-1}$; D *is a compound detour with* $Dh_{-1} = h_{+1}$; *and* $G_{+1}: [0, +1] \times S \to E$ *is a monotopy* $h_{+1} \sim g_{+1}$. (1) *If* D *is monotone and* $\operatorname{sgn}(D) = \operatorname{sgn}(G_i)$, $i = \pm 1$, *then* g_{-1} *is monotopic to* g_{+1} *via a modified monotopy* DG_{-1} *followed by* G_+. (2) *If* D *is mixed and* $\operatorname{sgn}(G_{-1}) \neq \operatorname{sgn}(G_{+1})$, *then* $D = D_{-1} + D_{+1}$, *each summand is monotone with* $\operatorname{sgn}(D_i) = \operatorname{sgn}(G_i)$, *and each* g_i *is monotopic to* $g_0 = D_{-1}h_{-1} = D_{+1}^* h_{+1}$ *via* $D_{-1}G_{-1}$, *resp.* $D_{+1}^* G_{+1}^*$.

Associated to $g \in L_N$ is a finite, totally ordered set $W(g)$, called the intersection sequence. For convenience, set $W = \{0, 1, \cdots, n\}$. It is obtained by setting $N_0 = g(\text{base point})$, and enumerating the nodes consecutively. Parametrizing S by $[0, 2\pi]$ allows the association to each $k \in W$ also $\{x_k', x_k''\} = g^{-1}(N_k)$, $0 \leqq x_k' < x_k'' < 2\pi$, and the kth subloop $g_k = g \mid [x_k', x_k'']$. For $i < j$ in W, there is a trichotomy of binary relations: $i \supset j$ if $x_i' < x_j' < x_j'' < x_i''$; $i \mid j$ if $x_i' < x_i'' < x_j' < x_j''$; iLj if $x_i' < x_j' < x_i'' < x_j''$. By abuse of language, relations predicated of indices are equally predicated of the corresponding nodes or subloops. Please see [1], [2] for details of this combinatorial description of L_N.

If the relation L is void in W, the sequence is properly nested. If both $|$ and L are void, W is chained. If $N_0 \in \text{Clos } \mathbf{C}_\infty(g)$ (=closure of the unbounded component of $E \backslash g(S)$), then g is said to start outside. The kth subloop is exterior if $\{g(x_k' - s), g(x_k'' + s) \mid s \text{ sufficiently small positive}\} \subset \mathbf{C}_\infty(g_k)$. In W is a canonical properly nested subsequence EW, of essential indices, obtained as follows: The initial index 0 is essential. If q is essential, $q_1 = \min\{j \mid q \supset j\}$ is essential; $q_{k+1} = \min\{j \mid q \supset j \text{ and } q_k \mid j\}$ is essential, $k = 1, 2, \cdots, m$. Let $[g/q] = g[x_q', x_{q(1)}'] \cup \bigcup_{1 \le k \le m-1} g[x_{q(k)}'', x_{q(k+1)}'] \cup g[x_{q(m)}'', x_q'']$. It is a Jordan loop.

PROPOSITION 4. *For $g \in L_N$, $EW(g)$ is properly nested. If $W(g)$ is already properly nested, $EW = W$. If g starts outside, every essential subloop is exterior. If g starts outside and $W(g)$ is not already properly nested, there is at least one essential subloop q that links (= there is a j with qLj or jLq), but no proper subloop of q links. All linking of q occurs on $[g/q]$ and q is linked an even number of times.*

For a normal loop g starting outside there is a sign computed for each k in $W(g)$ as follows: $\text{sgn}(0) = \pm 1$ according to which $\{g(0) \pm t\dot{g}(0) \mid \text{sufficiently small } t > 0\} \subset \mathbf{C}_\infty(g)$; $\text{sgn}(k) = \text{sgn} \det(g'(x_k''), g'(x_k'))$. A properly nested sequence W is precanonical if either $W = \{0, 1\}$ and $\text{sgn}(0) \ne \text{sgn}(1)$, or if all indices of W have the same sign. A precanonical sequence is canonical if it is also chained. Each tangent winding number class of normal loops has a unique canonical representative $W(g)$.

The various constructions are summed up by

PROPOSITION 5. *For $h \in L_N$, there is a mixed sum of simple detours U, so that $W(Uh) = EW(h)$. There is further, a mixed sum A, with $\text{supp}(A) \cap \text{supp}(U) = \varnothing$, so that $(U + A)h$ is precanonical. It is canonical for $TWN(h) = 0$ or ± 1. If $|TWN(h)| \ge 2$, there is a monotone sum B, $\text{sgn}(B) = \text{sgn}(TWN(h))$, $\text{supp}(B) \cap \text{supp}(U + A) = \varnothing$, so that $(U + A + B)h$ is canonical.*

PROPOSITION 6. *Let f_i, $i = \pm 1$, be canonical of like TWN. There are monotone compounds D_i, $\text{sgn}(D_i) = \text{sgn}(TWN + \frac{1}{2})$, so that $D_{-1}f_{-1}(S) = D_{+1}f_{+1}(S)$.*

Assembly of the proof of the theorem. Let g_i, $i = \pm 1$ be in L, of like TWN≥ 0. (The case TWN≤ 0 is essentially the mirror image.) By Proposition 1 there are positive monotopies G_i: $g_i \sim h_i$, $h_i \in L_N$. Apply

Propositions 5 and 6 to obtain canonical $f_i=(U_i+A_i+B_i)h_i$, and (after a suitable reparametrization of all objects indexed by $i=+1$) positive compounds D_i, so that $D_{-1}f_{-1}=D_{+1}f_{+1}$. The constructions were such that (for TWN$\geqq 2$) $\operatorname{supp}(D_i)\subset\operatorname{supp}(B_i)$, $\operatorname{sgn}(D_i)=\operatorname{sgn}(B_i)=+1$, and in all cases, $\operatorname{supp}(D_i)\cap\operatorname{supp}(U_i+A_i)=\emptyset$. Thus the compounds $U_i+A_i+D_iB_i$ are legitimate (read: $B_i=$identity for TWN$=0, 1$). Because U_i+A_i is a mixed sum, it can be reassociated to read M_i+N_i, M_i monotone positive, N_i monotone negative. Further, $\operatorname{supp}(N_i)\subset\operatorname{supp}(D_j)$, $j\neq i$, hence $N_i^*D_j$ is a legitimate positive product. Finally, it is legitimate to write

$$(M_{-1}+N_{+1}^*D_{-1}B_{-1})h_{-1}=(M_{+1}+N_{-1}^*D_{+1}B_{+1})h_{+1}=g_0.$$

The same argument as in Proposition 3 now completes the proof of the main Theorem, where $H_i=(M_i+N_j^*D_iB_i)G_i$, $i\neq j$.

References

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University of Michigan