CHARACTERIZATIONS OF THE ESSENTIAL SPECTRUM OF F. E. BROWDER

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Let T be a densely defined closed linear operator on a Banach space X. F. E. Browder [1] has defined the essential spectrum of T, $\mathrm{ess}(T)$, to be the set of complex numbers λ such that at least one of the following conditions is satisfied:

(i) The range $\mathfrak{R}(\lambda-T)$ of the operator $\lambda-T$ is not closed in X;

(ii) $\bigcup_{k\geq 0} \mathfrak{N}[(\lambda-T)^k]$ is of infinite dimension, ($\mathfrak{N}(S)$ being the null space of the operator S);

(iii) The point λ is a limit point of the spectrum of T.

In [7], M. Schechter discusses two other sets of complex numbers, $\sigma_{ew}(T)$ and $\sigma_{em}(T)$, which have also been called the essential spectrum of T (cf. [10]). He characterizes $\sigma_{em}(T)$ as the largest subset of the spectrum of T which remains invariant under compact perturbations of T. Although $\sigma_{em}(T)$ is in general a proper subset of $\mathrm{ess}(T)$, Schechter gives conditions which guarantee that $\mathrm{ess}(T)$ will remain invariant under compact (and certain other) perturbations of T. The proofs of these results usually reduce to showing that $\sigma_{em}(T)=\mathrm{ess}(T)$.

In this paper we replace Schechter's conditions on T by a condition on the perturbing operator and show that $\mathrm{ess}(T)$ is invariant under compact (and certain other) perturbations of T, provided the perturbing operators *commute* with T. We shall say that a linear operator C commutes with T if (i) the domain of C, $\mathfrak{D}(C)$, contains the domain of T, (ii) $Cx\in\mathfrak{D}(T)$ whenever $x\in\mathfrak{D}(T)$, (iii) and $TCx=CTx$ for $x\in\mathfrak{D}(T^2)$.

Following the notation and terminology of [9], we denote the dimension of the null space or *nullity* of an operator S by $n(S)$ and the codimension of the range or *defect* of S by $d(S)$. The *ascent* of S, $\alpha(S)$, is the smallest integer p such that $\mathfrak{N}(S^p)=\mathfrak{N}(S^{p+1})$, and the descent of S, $\delta(S)$, is the smallest integer q such that $\mathfrak{R}(S^q)=\mathfrak{R}(S^{q+1})$. (It may happen that $\alpha(S)=\infty$ or $\delta(S)=\infty$.) Suppose that λ_0 is a pole of order p of the resolvent operator $(\lambda-T)^{-1}$ and let E be the spectral projection corresponding to the spectral set $\{\lambda_0\}$. The range of E is the null space of $(\lambda_0-T)^p$ and the dimension of this space is called the *rank* of the pole λ_0.

THEOREM 1. *Let T be a densely defined closed linear operator on a*

Banach space X. Let λ_0 be a point of the spectrum of T. The following statements are equivalent:

(1) *λ_0 is not in* $\text{ess}(T)$.

(2) *λ_0 is a pole of the resolvent $(\lambda - T)^{-1}$ of finite rank.*

(3) *λ_0 has finite ascent, descent, and defect.*

(4) $n(\lambda_0 - T) = d(\lambda_0 - T) < \infty$ *and* $\alpha(\lambda_0 - T) < \infty$.

The equivalence of (1) and (2) was proved by F. E. Browder in Lemma 17 of [1]. Since that time results of Kaashoek [3], Taylor [9] and the author [6] have provided tools for giving a short proof of the equivalence of (1), (2), and (4) without requiring that T have a dense domain.

THEOREM 2. *Let T be a closed linear operator on a Banach space X. A point λ_0 in the spectrum of T is a pole of the resolvent of finite rank if and only if there is a compact linear operator C with $C(\mathfrak{D}(T)) \subset \mathfrak{D}(T)$ and $TCx = CTx$ for $x \in \mathfrak{D}(T^2)$ such that $\lambda_0 - (T + C)$ has a bounded inverse defined on all of X.*

COROLLARY. *Let T be a closed linear operator on a Banach space X. Then* $\text{ess}(T)$ *is the largest subset of the spectrum $\sigma(T)$ which remains invariant under perturbations of T by compact operators which commute with T, i.e.*

$$\text{ess}(T) = \{\lambda \mid \lambda \in \sigma(T + C) \textit{ for every compact operator } C \textit{ such that } C(\mathfrak{D}(T)) \subset \mathfrak{D}(T) \textit{ and } TCx = CTx \textit{ for } x \in \mathfrak{D}(T^2)\}.$$

Both $\sigma_{em}(T)$ and $\text{ess}(T)$ are also invariant under certain unbounded perturbations. Suppose that T is a closed linear operator in X and C is a linear operator with $\mathfrak{D}(C) \supset \mathfrak{D}(T)$. We say that C is *T-closable* if $x_n \to 0$, $Tx_n \to 0$, $Cx_n \to z$ for $\{x_n\} \subset \mathfrak{D}(T)$ implies $z = 0$. The operator C is *T-compact* if for any sequence $\{x_n\} \subset \mathfrak{D}(T)$ satisfying

$$\|x_n\| + \|Tx_n\| \leqq \text{const.},$$

the sequence $\{Cx_n\}$ has a convergent subsequence. The operator C is *T-pseudo-compact* if for any sequence $\{x_n\} \subset \mathfrak{D}(T)$ satisfying

$$\|x_n\| + \|Tx_n\| + \|Cx_n\| \leqq \text{const.},$$

the sequence $\{Cx_n\}$ has a convergent subsequence. Theorem 3 below is the analogue of Theorems 2.1 and 2.2 of [7] (although in that paper T and C were densely defined).

THEOREM 3. *Let T be a closed linear operator on a Banach space X. Then* $\text{ess}(T)$ *is the largest subset of the spectrum of T which remains in-*

variant under perturbations of T by operators C which commute with T and are either T-compact or are T-closable and T-pseudo-compact.

The compactness condition on the perturbing operator C may be generalized in another direction. It is well known that

$$n(T) - d(T) = n(T + C) - d(T + C)$$

when C is a strictly singular operator (or is strictly singular relative to T, cf. [4]), or C is an inessential operator (cf. [5]). From Schechter's characterization [7] of $\sigma_{em}(T)$ as the complement in the complex plane of the set of points λ for which $n(\lambda - T) = d(\lambda - T) < \infty$ it follows immediately that $\sigma_{em}(T)$ is the largest subset of the spectrum which remains invariant under perturbations of T by strictly singular or inessential operators. Even more is true in the analogous situation for $\text{ess}(T)$. The ideal of strictly singular operators and the ideal of inessential operators are both contained in a set of bounded linear operator called Riesz operators [2].

A Riesz operator R is characterized by the property that it is a bounded linear operator with $d(\lambda - R) < \infty$ for all $\lambda \neq 0$ [6].

THEOREM 4. *Let T be a closed linear operator on a Banach space X. Then* $\text{ess}(T)$ *is the largest subset of the spectrum of T which remains invariant under perturbations of T by Riesz operators R which commute with T.*

Proofs of these results will appear elsewhere.

REFERENCES

1. F. E. Browder, *On the spectral theory of elliptic differential operators*. I, Math. Ann. **142** (1961), 22–130.

2. S. R. Caradus, *Operators of Riesz type*, Pacific J. Math. **18** (1966), 61–71.

3. M. A. Kaashoek, *Ascent, descent, nullity and defect, a note on a paper by A. E. Taylor*, Math. Ann. **172** (1967), 105–115.

4. T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Analyse Math. **6** (1958), 261–322.

5. D. Kleinecke, *Almost-finite, compact and inessential operators*, Proc. Amer. Math. Soc. **14** (1963), 863–868.

6. D. C. Lay, *Studies in spectral theory using ascent, descent, nullity and defect*, Doctoral dissertation, University of California, Los Angeles, 1966.

7. M. Schechter, *On the essential spectrum of an arbitrary operator*. I, J. Math. Anal. Appl. **13** (1966), 205–215.

8. A. E. Taylor, *Introduction to functional analysis*, Wiley and Sons, New York, 1958.

9. ———, *Theorems on ascent, descent, nullity and defect of linear operators*, Math. Ann. **163** (1966), 18–49.

10. F. Wolf, *On the essential spectrum of partial differential boundary problems*, Comm. Pure Appl. Math. **12** (1959), 211–228.

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