CONJUGATE LOCI IN GRASSMANN MANIFOLDS

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1. **Introduction.** In the tangent space M_x to a Riemannian manifold M at the point x, a conjugate point v is a point at which the differential of the exponential map $\exp_x: M_x \to M$ is singular. In M, a point y is a conjugate point to x if $y = \exp_x v$ for some conjugate point v in M_x. The conjugate locus in M_x is the set of conjugate points in M_x, and the conjugate locus in M at x is the set of conjugate points to x.

Though there are a number of general results on the conjugate locus either in M_x or in M ([4], [6, p. 59], [11], [12] and [13]), the precise nature of this locus in special Riemannian manifolds seems to be known only in a few cases, such as the sphere, the projective spaces, and some two-dimensional manifolds ([2, pp. 225–226], [9] and [10]). In the present note, we give a complete description of the conjugate locus at a point in the real, complex or quaternionic Grassmann manifolds. Besides being useful and interesting, this information will extend the range of problems recently studied by Klingenberg [8], Allamigeon [1], Green [5] and Warner [12, 13]. The conjugate locus in the tangent space to a Grassmann manifold is more complex and will be the subject of a future note.

In §2, we describe the Schubert varieties of which the conjugate locus in a Grassmann manifold is composed. In §3, we give some results concerning conjugate points in a Grassmann manifold. In §4, we state our main theorem. Details and proof will be omitted. For background information, the reader is referred to the author's paper [14].

2. **Some Schubert varieties** (cf. [3, Chapter 4] and [7, Chapter 14]). Let F be the field R of real numbers, the field C of complex numbers, or the field H of real quaternions; F^{n+m} an $(n+m)$-dimensional left vector space over F provided with a positive definite hermitian inner product; $G_n(F^{n+m})$ the Grassmann manifold of n-planes in F^{n+m}.

In F^{n+m}, let P be a fixed p-plane $(1 < p < n+m)$, Z a variable n-plane, and

$$V_l = \{Z: \dim(Z \cap P) \geqq l\} \qquad (l \geqq 0),$$

$$W_l = V_l \backslash V_{l+1} = \{Z: \dim(Z \cap P) = l\} \qquad (l \geqq 0).$$

Then it is easy to see that $V_l = G_n(F^{n+m})$ if $l = \max(0, p-m)$, and V_l is empty if $l > \min(n, p)$. For the remaining values of l, we can prove

THEOREM 2.1. *Let k be any integer such that*

$$\max(1, p - m + 1) \leqq k \leqq \min(n, p).$$

(a) *The subset V_k of $G_n(F^{n+m})$ is a Schubert variety*

$$(p - k, \cdots, p - k, m, \cdots, m),$$

where $p-k$ appears k times and m appears $n-k$ times. The F-dimension of V_k is $nm-k(m-p+k)$.

(b) *V_{k+1} is the singular locus of V_k.*

(c) *V_k can be decomposed into the disjoint union*

$$V_k = W_k \cup W_{k+1} \cup \cdots \cup W_{\min(n,p)},$$

where

$$\begin{aligned} W_{\min(n,p)} &= G_n(F^p) && \text{if } p > n, \\ &= \{P\} && \text{if } p = n, \\ &\approx G_m(F^{n+m-p}) && \text{if } p < n, \end{aligned}$$

and each W_l $(k \leqq l \leqq \min(n, p)-1)$ is a "tensor" bundle whose base space is $G_l(F^p) \times G_{n-l}(F^{n+m-p})$, whose standard fiber is the tensor product $(F^{n-l})^* \otimes F^{p-l}$ of an $(n-l)$-dimensional right vector space and a $(p-l)$ dimensional left vector space, and whose group is the tensor product $GL(n-l, F) \otimes GL(p-l, F)$; the fiber of W_l over the point $(\mathbf{x}, \mathbf{y}) \in G_l(F^p) \times G_{n-l}(F^{n+m-p})$ consists of all those n-planes $\mathbf{Z}$ such that $\mathbf{Z} \cap \mathbf{P}$ is the fixed l-plane $\mathbf{x}$, and the projection of $\mathbf{Z}$ in the orthogonal complement of $\mathbf{P}$ in F^{n+m} is the fixed $(n-l)$-plane $\mathbf{y}$.

Two special cases are of interest to us. Let $\mathbf{O}$ be a fixed n-plane in F^{n+m} and $\mathbf{O}^\perp$ its orthogonal complement, and let

$$V_l = \{\mathbf{Z}: \dim(\mathbf{Z} \cap \mathbf{O}^\perp) \geqq l\}, \qquad \tilde{V}_l = \{\mathbf{Z}: \dim(\mathbf{Z} \cap \mathbf{O}) \geqq l\}.$$

It turns out that the cut locus at the point $\mathbf{O}$ in $G_n(F^{n+m})$ is V_1 (see [**14**, Theorem 9(b)]), and the conjugate locus at the point $\mathbf{O}$ in $G_n(F^{n+m})$ is the union of V_1 or V_2 and one of the $\tilde{V}_l$'s (see §4).

3. **Geodesics and conjugate points in $G_n(F^{n+m})$.** As in [**14**], let $G_n(F^{n+m})$ be provided with the invariant Riemannian metric $ds^2 = \sum_i (d\theta_i)^2$, where $d\theta_i$ $(1 \leqq i \leqq n)$ are the n angles between two consecutive n-planes in F^{n+m}. Then $G_n(F^{n+m})$ is a complete globally-

symmetric space. It is known that the geodesics in $G_n(F^{n+m})$, when viewed as a 1-parameter family of n-planes in F^{n+m}, are characterized by the following properties: (a) All the pairs of nearby n-planes of this family have common angle 2-planes (some of which may degenerate into angle 1-planes), and (b) the n angles between every pair of nearby n-planes are proportional to a fixed set of constants.

Let O and A be any two points in $G_n(F^{n+m})$, and Γ any geodesic segment joining O and A. Then each common angle 2-plane of Γ either coincides with or contains an angle 2-plane between O and A. If q (resp. p) is the number of nondegenerate angle 2-planes of Γ (resp. between O and A), so that $1 \leqq p \leqq q \leqq \min(n, m)$, then Γ is said to be of *the* $(q-p+1)$*th type*. Among the geodesic segments of the $(q-p+1)$th type joining O and A, the shortest ones are of length

$$[(\theta_1)^2 + \cdots + (\theta_p)^2 + (q-p)\pi^2]^{1/2},$$

where $\theta_1, \cdots, \theta_p$ are the nonzero angles between O and A. Such a geodesic segment is called a *minimal geodesic segment of the* $(q-p+1)$*th type*. Obviously, a minimal geodesic segment of the kth type is shorter than one of the $(k+1)$th type. A minimal geodesic segment of the first type is a minimal segment in the usual sense.

We can prove

THEOREM 3.1. *In* $G_n(F^{n+m})$, *a point* A *is a conjugate point to the point* O *iff there exists a continuous family of (distinct) minimal geodesic segments of the first or the second type joining* O *and* A.

Concerning first conjugate points, we can prove

THEOREM 3.2. (a) *In a* $G_n(R^{n+m})$, *any conjugate point* A *to the point* O *is the first conjugate point to* O *along some minimal geodesic segment of the first or the second type joining* O *and* A.

(b) *In a* $G_n(C^{n+m})$ *or* $G_n(H^{n+m})$, *a conjugate point* A *to the point* O *either is the first conjugate point to* O *along some minimal segment joining* O *and* A, *or is such that the mid-point of some minimal geodesic segment* Γ *of the second type joining* O *and* A *is the first conjugate point to* O *along* Γ.

Given two points O and A in $G_n(F^{n+m})$, the existence or nonexistence of a continuous family of minimal geodesic segments of the first or the second type joining them and the nature of such a family if it exists depend entirely on the field F and the dimensions of $A \cap O^\perp$ and $A \cap O$. A study of the various possibilities leads to our next theorem. We first give a definition.

A conjugate point A to the point O in $G_n(F^{n+m})$ is said to be of *type-order* $[k, h]$ if there exists a maximal continuous (then C^ω) family of ∞^h $(h \geqq 1)$ minimal geodesic segments of the kth type joining O and A, and k is the smallest integer possible.

It follows from Theorem 3.1 that $k=1$ or 2.

4. **Conjugate locus in $G_n(F^{n+m})$.** Let O be any n-plane in F^{n+m}; V_l, $\tilde{V}_l$ as defined in the last paragraph of §2; and $W_l = V_l \backslash V_{l+1}$, $\tilde{W}_l = \tilde{V}_l \backslash \tilde{V}_{l+1}$. Then O is a point of $G_n(F^{n+m})$, and we have

THEOREM 4.1. (a) *In a $G_n(R^{n+m})$, the conjugate locus at the point O is*

$$V_2 \cup \tilde{V}_1 = (W_2 \cup W_3 \cup \cdots \cup W_n) \cup (\tilde{W}_1 \cup \tilde{W}_2 \cup \cdots \cup \tilde{W}_n) \quad \text{if } n<m;$$

$$V_2 \cup \tilde{V}_2 = (W_2 \cup W_3 \cup \cdots \cup W_n) \cup (\tilde{W}_2 \cup \tilde{W}_3 \cup \cdots \cup \tilde{W}_n) \quad \text{if } n=m;$$

$$V_2 \cup \tilde{V}_{n-m+1} = (W_2 \cup W_3 \cup \cdots \cup W_m) \cup (\tilde{W}_{n-m+1} \cup \tilde{W}_{n-m+2} \cup \cdots \cup \tilde{W}_n) \quad \text{if } n>m.$$

Points of W_l and $\tilde{W}_l \backslash V_2$ are conjugate points to O of type-order $[1, \frac{1}{2}l(l-1)]$ *and* $[2, m-n+2(l-1)]$, *respectively.*

(b) *In a $G_n(C^{n+m})$, the conjugate locus at the point O is*

$$V_1 \cup \tilde{V}_1 = (W_1 \cup W_2 \cup \cdots \cup W_n) \cup (\tilde{W}_1 \cup \tilde{W}_2 \cup \cdots \cup \tilde{W}_n) \quad \text{if } n \leqq m;$$

$$V_1 \cup \tilde{V}_{n-m+1} = (W_1 \cup W_2 \cup \cdots \cup W_m) \cup (\tilde{W}_{n-m+1} \cup \tilde{W}_{n-m+2} \cup \cdots \cup \tilde{W}_n) \quad \text{if } n>m.$$

Points of W_l and $\tilde{W}_l \backslash V_1$ are conjugate points to O of type-order $[1, l^2]$ *and* $[2, 2(m-n+2l)-3]$, *respectively.*

(c) *In a $G_n(H^{n+m})$, the conjugate locus at the point O is*

$$V_1 \cup \tilde{V}_1 = (W_1 \cup W_2 \cup \cdots \cup W_n) \cup (\tilde{W}_1 \cup \tilde{W}_2 \cup \cdots \cup \tilde{W}_n) \quad \text{if } n \leqq m;$$

$$V_1 \cup \tilde{V}_{n-m+1} = (W_1 \cup W_2 \cup \cdots \cup W_m) \cup (\tilde{W}_{n-m+1} \cup \tilde{W}_{n-m+2} \cup \cdots \cup \tilde{W}_n) \quad \text{if } n>m.$$

Points of W_l and $\tilde{W}_l \backslash V_1$ are conjugate points to O of type-order $[1, l(2l+1)]$ *and* $[2, 4(m-n+2l)-5]$, *respectively.*

Theorem 3.2 shows that in a $G_n(R^{n+m})$ the first conjugate locus coincides with the conjugate locus. It is known [14] that the minimum (or cut) locus in any $G_n(F^{n+m})$ at the point O is V_1. Thus it follows from Theorems 3.2 and 4.1 that in a $G_n(C^{n+m})$ or $G_n(H^{n+m})$ the first conjugate locus coincides with the minimum locus. This is a special case of a known result due to Crittenden [4, Theorem 5].

We conclude with two special cases of Theorem 4.1.

(1) *The projective spaces* $FP^m = G_1(F^{m+1})$, $m \geqq 1$. In this case, we have the following results, already known (see, for example, [2, pp. 225–226]):

	Conjugate locus at 0	Type-order of conjugate point
RP^1	Empty	
RP^m, $m>1$	$\{O\}$	$\{O\}$: $[2, m-1]$
CP^m, $m \geqq 1$	$O^{\perp} \cup \{O\}$	$O^{\perp}$: $[1, 1]$, $\{O\}$: $[2, 2m-1]$
HP^m, $m \geqq 1$	$O^{\perp} \cup \{O\}$	$O^{\perp}$: $[1, 3]$, $\{O\}$: $[2, 4m-1]$

(2) *The* $G_2(F^{m+2})$, $m \geqq 2$. In this case, we have

	Conjugate locus at 0	Type-order of conjugate point
$G_2(R^4)$	$\{O^{\perp}\} \cup \{O\}$	$\{O^{\perp}\}$: $[1, 1]$, $\{O\}$: $[2, 2]$
$G_2(R^{m+2})$, $m>2$	$W_2 \cup \tilde{W}_1 \cup \{O\}$	W_2: $[1, 1]$, $\tilde{W}_1$: $[2, m-2]$, $\{O\}$: $[2, m]$
$G_2(C^{m+2})$, $m \geqq 2$	$W_1 \cup W_2 \cup \tilde{W}_1 \cup \{O\}$	W_1: $[1, 1]$, W_2: $[1, 4]$ $\tilde{W}_1 \backslash W_1$: $[2, 2m-3]$, $\{O\}$: $[2, 2m+1]$
$G_2(H^{m+2})$, $m \geqq 2$	$W_1 \cup W_2 \cup \tilde{W}_1 \cup \{O\}$	W_1: $[1, 3]$, W_2: $[1, 10]$ $\tilde{W}_1 \backslash W_1$: $[2, 4m-5]$, $\{O\}$: $[2, 4m+3]$

where

$$W_1 = \{Z: \dim(Z \cap O^{\perp}) = 1\}, \qquad \tilde{W}_1 = \{Z: \dim(Z \cap O) = 1\}$$

are respectively an "$(m-1)$-plane" bundle and a "line" bundle over $W_1 \cap \tilde{W}_1 = FP^1 \times FP^{m-1}$, and

$$\begin{aligned} W_2 = \{Z: \dim(Z \cap O^{\perp}) = 2\} &= \{O^{\perp}\} \quad \text{if } m = 2, \\ &= G_2(F^m) \quad \text{if } m > 2. \end{aligned}$$

References

1. A. Allamigeon, *Propriétés globales des espaces de Riemann harmoniques*, Ann. Inst. Fourier, Grenoble (15) 2 (1965), 91–132.

2. R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.

3. S. S. Chern, *Topics in differential geometry*, Institute for Advanced Study lecture notes, Princeton, N. J., 1951.

4. R. Crittenden, *Minimum and conjugate points in symmetric spaces*, Canad. J. Math. 14 (1962), 320–328.

5. L. W. Green, *Auf Wiedersehenflächen*, Ann. of Math. **78** (1963), 289–299.

6. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

7. W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry*, Vol. II, Cambridge Univ. Press, New York, 1952.

8. W. Klingenberg, *Manifolds with restricted conjugate locus*, Ann. of Math. **78** (1963), 527–547.

9. S. Myers, *Connections between differential geometry and topology*, I. *Simply connected surfaces*, Duke Math. J. **1** (1935), 376–391.

10. ———, *Connections between differential geometry and topology*, II. *Closed surfaces*, Duke Math. J. **2** (1936), 95–102.

11. H. E. Rauch, *Geodesics and Jacobi equations on homogeneous Riemannian manifolds*, Proceedings of U.S.-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965, Tokyo (1966), pp. 115–127.

12. F. W. Warner, *The conjugate locus of a Riemannian manifold*, Amer. J. Math. **87** (1965), 575–604.

13. ———, *Conjugate loci of constant order*, Ann. of Math. **86** (1967), 192–212.

14. Y. C. Wong, *Differential geometry of Grassmann manifolds*, Proc. Nat. Acad. Sci. U.S.A. **57** (1967), 589–594.

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