A GAME WITH NO SOLUTION[1]

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1. Introduction. In 1944 von Neumann and Morgenstern [2] introduced a theory of solutions for n-person games in characteristic function form. The main mathematical question concerning their model is whether every game has at least one solution. This announcement describes a ten-person game which has no solution. The essential definitions for an n-person game will be reviewed briefly before the particular example is given. The proof that the game has no solution will then be sketched; a detailed proof will be published elsewhere.

2. **Definitions.** An *n-person game* is a pair (N, v) where $N = \{1, 2, \cdots, n\}$ is the set of players and v is a characteristic function on 2^N, i.e., v assigns the real number $v(S)$ to each subset S of N and $v(\phi)=0$. The set of *imputations* is

$$A = \left\{x\colon \sum_{i\in N} x_i = v(N) \text{ and } x_i \geqq v(\{i\}) \text{ for all } i \in N\right\}$$

where $x=(x_1, x_2, \cdots, x_n)$ is a vector with real components. For any $X\subset A$ and nonempty $S\subset N$, define $\text{Dom}_S X$ to be the set of all $x\in A$ such that there exists a $y\in X$ with $y_i > x_i$ for all $i\in S$ and with $\sum_{i\in S} y_i \leqq v(S)$. Let $\text{Dom}\, X = \bigcup_{S\subset N} \text{Dom}_S X$. Also let $\text{Dom}^{-1}X$ be the set of all $y\in A$ such that there exists $x\in X$ with $x\in \text{Dom}\{y\}$. A subset K of A is a *solution* if $K\cap \text{Dom}\, K=\phi$ and $K\cup \text{Dom}\, K=A$. If $X\subset A$ and $K'\subset X$, then K' is a *solution for* X if $K'\cap\text{Dom}\, K'=\phi$ and $K'\cup\text{Dom}\, K'\supset X$. The *core* of a game is

$$C = \left\{x \in A\colon \sum_{i\in S} x_i \geqq v(S) \text{ for all } S \subset N\right\}.$$

For any solution K, $C\subset K$ and $K\cap\text{Dom}\, C=\phi$.

A characteristic function v is *superadditive* if $v(S_1\cup S_2)\geqq v(S_1)+v(S_2)$ whenever $S_1\cap S_2=\phi$. The game listed below does not have a superadditive v as assumed in the classical theory. However, it is equivalent solutionwise to a game with a superadditive v. (See Gillies [1, p. 68].)

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3. **Example.** Consider the game (N, v) where $N=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and v is given by:

$$v(N) = 5, \ v(\{1, 3, 5, 7, 9\}) = 4,$$
$$v(\{1, 2\}) = v(\{3, 4\}) = v(\{5, 6\}) = v(\{7, 8\}) = v(\{9, 10\}) = 1,$$
$$v(\{3, 5, 7, 9\}) = v(\{1, 5, 7, 9\}) = v(\{1, 3, 7, 9\}) = 3,$$
$$v(\{3, 5, 7\}) = v(\{1, 5, 7\}) = v(\{1, 3, 7\}) = 2,$$
$$v(\{3, 5, 9\}) = v(\{1, 5, 9\}) = v(\{1, 3, 9\}) = 2,$$
$$v(\{1, 4, 7, 9\}) = v(\{3, 6, 7, 9\}) = v(\{5, 2, 7, 9\}) = 2,$$
$$v(S) = 0 \quad \text{for all other } S \subset N.$$

For this game

$$A = \left\{x: \sum_{i\in N} x_i = 5 \text{ and } x_i \geqq 0 \text{ for all } i \in N\right\}.$$

One can also show that C is the convex hull of the six imputations:

(1,0,1,0,1,0,1,0,1,0), (0,1,1,0,1,0,1,0,1,0), (1,0,0,1,1,0,1,0,1,0), (1,0,1,0,0,1,1,0,1,0), (1,0,1,0,1,0,0,1,1,0), and (1,0,1,0,1,0,1,0,0,1).

4. **Outline of proof.** Consider the following subsets of A:

$$B = \{x \in A: x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = x_9 + x_{10} = 1\},$$
$$E_i = \{x \in B: x_j = x_k = 1, x_i < 1, x_7 + x_9 < 1\},$$
$$E = \cup_i E_i, \qquad i = 1, 3, 5,$$
$$\begin{aligned} F = \Big[&\cup_{(j,k)} \{x \in B: x_j = x_k = 1, x_7 + x_9 \geqq 1\} \\ &\cup \cup_{(p,q)} \{x \in B: x_p = 1, x_q < 1, x_3 + x_5 + x_q \geqq 2, \\ &\qquad x_1 + x_5 + x_q \geqq 2, x_1 + x_3 + x_q \geqq 2\} \\ &\cup \{x \in B: x_7 = x_9 = 1\} \cup \{x \in B: x_1 = x_3 = x_5 = 1\}\Big] - C, \end{aligned}$$

where $(i, j, k) = (1, 3, 5)$, $(3, 5, 1)$, and $(5, 1, 3)$; and $(p, q) = (7, 9)$ and $(9, 7)$. One can verify that the subsets $A-B$, $B-(C\cup E\cup F)$, C, E, and F form a partition of A.

To prove that this game has no solution it is sufficient to prove that

(1) Dom $C \supset [A-B] \cup [B-(C\cup E\cup F)]$,

(2) $E \cap$ Dom $(C\cup F) = \phi$, and

(3) there is no solution for E.

One can prove (1) and (2) by checking various subsets S of N. In fact, one can prove in addition that Dom $C = A-(C\cup E\cup F)$, and

$F \cap \text{Dom}(C \cup E \cup F) = \phi$; and thus $C \cup F$ is contained in every solution.

Now consider the region E. One can check that $E_i \cap \text{Dom}_S E = \phi$ for all S except $\{i, r, 7, 9\}$, and

$$E_i \cap \text{Dom}_{\{i,r,7,9\}}(E_i \cup E_k) = \phi$$

where $(i, r, k) = (1, 4, 5)$, $(3, 6, 1)$, and $(5, 2, 3)$. Thus the "Dom" pattern in E is cyclic as illustrated by the diagram:

$$E_5 \underset{\{3,6,7,9\}}{\rightarrow} E_3 \underset{\{1,4,7,9\}}{\rightarrow} E_1 \underset{\{5,2,7,9\}}{\rightarrow} E_5.$$

To prove (3), assume that $K'(\neq \phi)$ is a solution for E and pick any $y \in K'$. Using the symmetry in E, one can assume $y \in E_3$. Define

$$G_i(y) = \{x \in E_i: x_7 > y_7, x_9 > y_9, x_k + x_r + x_7 + x_9 \leqq 2\}$$

where $(i, k, r) = (1, 5, 2)$, $(3, 1, 4)$, and $(5, 3, 6)$. Then one can verify that $E \cap \text{Dom}^{-1}\{y\} = G_5(y)$, and so $K' \cap G_5(y) = \phi$. However, $E \cap \text{Dom}^{-1} G_5(y) = G_1(y)$, and so

$$K' \cap G_1(y) \neq \phi.$$

On the other hand, $G_3(y) \cap \text{Dom}(E_5 - G_5(y)) = \phi$, and so $G_3(y) \subset K'$. However, $G_1(y) \subset \text{Dom } G_3(y)$, and so

$$K' \cap G_1(y) = \phi$$

which gives a contradiction. Therefore, there is no solution K' for E.

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References

1. D. B. Gillies, "Solutions to general non-zero-sum games," *Annals of Mathematics Studies*, No. 40, A. W. Tucker and R. D. Luce (eds.), Princeton Univ. Press, Princeton, N. J., 1959, pp. 47–85.

2. J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton Univ. Press, Princeton, N. J., 1944.

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