NONNEGATIVE EIGEN FUNCTIONS OF LAPLACE-BELTRAMI OPERATORS ON SYMMETRIC SPACES

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1. **Introduction.** Let G be a connected semisimple Lie group with a finite center and let K be a maximal compact subgroup of G and let $X=G/K$ be the homogeneous space of left cosets $gK(g\in G)$ of the group G with respect to the subgroup K. Then it is known that a G-invariant Riemannian metric can be introduced in the space X so that X becomes a Riemannian symmetric space of nonpositive curvature. Let $\mathfrak{G}$ be the Lie algebra of G and let $\mathfrak{K}$ be the subalgebra of $\mathfrak{G}$ corresponding to the subgroup K. Let $\mathfrak{P}$ be the orthogonal complement of $\mathfrak{K}$ in $\mathfrak{G}$ with respect to the Killing form $\langle\ ,\ \rangle$ of the algebra $\mathfrak{G}$. Let $\mathfrak{A}$ be a maximal abelian subspace of $\mathfrak{P}$. Then $\mathfrak{A}$ is a Cartan subalgebra of the symmetric space X. Let $\mathfrak{A}'$ be the set of all regular elements in $\mathfrak{A}$ and let $\mathfrak{A}^+$ be a fixed component (connected) in $\mathfrak{A}'$. Then the set $\mathfrak{A}^+$ is a Weyl chamber in the space $\mathfrak{A}$. Let $\mathfrak{A}^*$ be the dual space of the space $\mathfrak{A}$. Then the space $\mathfrak{A}^*$ can be identified with the space $\mathfrak{A}$ by means of the Killing form $\langle ,\rangle$ of $\mathfrak{G}$ in the usual manner. Let $\alpha\in\mathfrak{A}^*$. We set

$$\mathfrak{G}_\alpha = \{X \in \mathfrak{G}\colon [H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{A}\},$$

$d_\alpha=\dim \mathfrak{G}_\alpha$. Then α is said to be a root of the space X with respect to the Cartan subalgebra $\mathfrak{A}$, if $d_\alpha>0$. A root α is said to be positive if $\alpha(H)>0$ for all $H\in\mathfrak{A}^+$. Let P be the set of all positive roots of X with respect to $\mathfrak{A}$. We set

$$\rho = \tfrac{1}{2}\sum_{\alpha\in P} d_\alpha\alpha; \qquad \mathfrak{N} = \sum_{\alpha\in P}\mathfrak{G}_\alpha$$

$$A = \exp(\mathfrak{A}); \qquad N = \exp(\mathfrak{N}).$$

Then we have the Iwasawa decomposition: $G=KAN$ where A and N are connected commutative and nilpotent subgroups of G respectively.

Let $a\in A$. Then there exists a unique element $H\in\mathfrak{A}$ such that $a=\exp H$. We then write $H=\ln a$.

2. **Some basic prerequisites.** We now give some results from [4]

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which are instrumental for the formulation of the main theorems in the next section.

A horocycle ξ in X is an orbit in the space X of a group of the form gNg^{-1}, $g\in G$. Let Ξ be the set of all horocycles in X. Let M be the centralizer of A in K. Then the set Ξ can be identified with the homogeneous space G/MN. Moreover the homogeneous space K/M can be identified with the set of all Weyl chambers contained in all maximal abelian subspaces of the space $\mathfrak{P}$.

Let $x_0=\{K\}$ be the origin in X and let $\xi_0=N\cdot x_0$ be the origin in Ξ. Let $\xi\in\Xi$ be arbitrary. Then the horocycle ξ can be written as

$$\xi = ka\xi_0$$

where $a\in A$ is unique and $k\in K$ is unique (mod M). Here the Weyl chamber kM is said to be normal to the horocycle ξ and the element $a\in A$ is said to be the complex distance from x_0 to ξ. We set $B=K/M$. There the space B can be identified with the Furstenberg boundary $B(G)$ (cf. [2]).

Let $x\in X$, $b\in B$. Let $\xi(x, b)$ be the horocycle with normal b and passing through the point x. Let $a(x, b)$ be the complex distance from x_0 to $\xi(x, b)$. Let $H(x, b)=\ln a(x, b)$ so that $H(x, b)\in\mathfrak{A}$. It is shown in [4] that the elements $H(x, b)$ play an essential role in defining the spherical Fourier transform of an arbitrary function $f\in C_c^\infty(X)$.

3. **Main results.** Let Δ be the Laplace-Beltrami operator in the space X and let c be some real number. We first give a complete description of the cone of all nonnegative solutions of the equation

$$\Delta f = cf. \tag{1}$$

It is shown in [5] that, for $c<-\langle\rho, \rho\rangle$, the equation (1) does not have a nonnegative solution (except the trivial solution $f=0$). Hence we consider only the case $c\geqq -\langle\rho, \rho\rangle$.

A nonnegative solution f of (1) is said to be normalized if $f(x_0)=1$. A nonnegative solution f of (1) is said to be minimal, if every nonnegative solution of (1) which does not exceed f is a constant multiple of f.

Let $c\geqq -\langle\rho, \rho\rangle$. We set

$$\mathfrak{A}_c = \{H\in\mathfrak{A}: \langle H, H\rangle = c+\langle\rho, \rho\rangle\} \qquad \text{and} \qquad \mathfrak{A}_c^+ = \mathfrak{A}^+\cap\mathfrak{A}_c.$$

Let $b\in B$, $\lambda\in\mathfrak{A}_c^+$. We now define the function $\phi_{b,\lambda}$ on X by the formula

$$\phi_{b,\lambda}(x) = e^{(\lambda+\rho)(H(x,b))} \qquad (x\in X). \tag{2}$$

THEOREM 1. *The set of all normalized minimal solutions of the equation* (1) *coincide with the set of all functions*

$$\{\phi_{b,\lambda}: b \in B, \lambda \in \mathfrak{A}_c^+\}.$$

The proof can be carried out by using the method of induction on the rank of the symmetric space X as in [5].

THEOREM 2. *A function f is a nonnegative solution of* (1) *if and only if f can be represented in the form*

$$f(x) = \int_{B\times\mathfrak{A}_c^+} \phi_{b,\lambda}(x)d\mu(b, \lambda). \tag{3}$$

Here μ is a finite positive Radon measure on $B\times\mathfrak{A}_c^+$ which is uniquely determined by f.

The proof is an immediate consequence of Theorem 1 and Choquet's Theorem (cf. [1]).

Let $\mathfrak{D}(X)$ be the algebra of all G-invariant differential operators on X. Then a function $f\in C^\infty(X)$ is said to be semi-spherical, if f is an eigen function of every differential operator $D\in\mathfrak{D}(X)$ and moreover satisfies the relation $f(x_0)=1$. Clearly a semi-spherical function f is a spherical function on X if and only if f is K-invariant.

THEOREM 3. *A function f on X is a nonnegative semi-spherical function on X if and only if f can be represented in the form*

$$f(x) = \int_B \phi_{b,\lambda}(x)d\mu(b). \tag{4}$$

Here $\lambda\in\mathfrak{A}^+$ and μ is a finite positive Radon measure on B such that $\int_B d\mu(b)=1$. Moreover the pair (μ, λ) is determined uniquely by the function f.

THEOREM 4. *A function f on X is a nonnegative spherical function on X if and only if f can be represented in the form*

$$f(x) = \int_B \phi_{b,\lambda}(x)db. \tag{5}$$

Here $\lambda\in\mathfrak{A}^+$ and db is the unique K-invariant positive measure on B such that $\int_B db=1$.

REMARK 1. This theorem can be considered as a special case of a more general result of Harish-Chandra [3] which gives the integral representation of an arbitrary complex-valued spherical function on X.

A function f on X is said to be harmonic, if f satisfies the equation $\Delta f=0$.

THEOREM 5. *A function f on X is a bounded nonnegative harmonic function on X if and only if f can be represented in the form*

$$f(x) = \int_B e^{2\rho(H(x,b))}\hat{f}(b)\,db. \tag{6}$$

Here $\hat{f}$ is a bounded nonnegative measurable function on B and is determined uniquely by f almost everywhere on B.

REMARK 2. Here the function $e^{2\rho(H(x,b))}$ is the Poisson kernel and the formula (6) is the analog of Poisson integral formula for bounded nonnegative harmonic functions on symmetric spaces (cf. [2]).

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