REPRESENTATIONS OF LOCALLY FINITE GROUPS

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Communicated by G. B. Seligman, August 15, 1967

The purpose of this paper is to give a brief general account of the completely reducible finite-dimensional representations of a locally finite group G over a given algebraically closed field K. Theorem 1 shows that all such representations of G can be brought down to the algebraic closure F in K of the prime field of K. This reduces all further considerations in this account to countable groups. Theorem 2 characterizes the existence of a faithful completely reducible representation of G of degree n over K in terms of the existence of such representations for appropriate finite subgroups of G.

Throughout the paper, G denotes a locally finite group, K denotes an arbitrary algebraically closed field and F denotes the algebraic closure in K of the prime field of K. V denotes an n-dimensional vector space over K. An *F-form of V* is an F-subspace W of V such that W and K are linearly disjoint over K and V is the K-span of W. (Equivalently, an F-form of V is the F-span of a basis of V.) If A is an F-algebra, A_K denotes the algebra $A \otimes_F K$.

THEOREM 1. *Let ρ be a completely reducible representation of G in V. Then V has an F-form W which is stable under the ρ-action of G in V.*

PROOF. It suffices by complete reducibility to consider the case in which G acts irreducibly in V.

If G is finite, the assertion follows (upon passing to the group algebra of G over F) from the fact that if A is a finite-dimensional associative algebra over F, then an irreducible (finite-dimensional) A_K-module has an F-form stable under A. Since the kernel of an irreducible representation of such an A contains the radical of A, it suffices to prove this in the case that A is semisimple. But for A semisimple, the assertion is obvious since:

(1) $A = \sum_1^m \oplus A_i$ where $A_1, \cdots, A_m$ are minimal right ideals of A;
(2) $A_K = \sum_1^m \oplus (A_i)_K$ and the $(A_i)_K$ are minimal right ideals of A_K;
(3) any irreducible A_K-module is isomorphic to one of the $(A_i)_K$ [1, Chapter IV].

Next assume that G is locally finite and that (ρ, V) is an irreducible representation of G over K of degree n. Then some finite subgroup H of G acts irreducibly in V. (For example let S be a finite subset of G

[1] This research was partially supported by research grant NSF-GP-4017.

such that $\rho(S)$ is a maximal K-independent subset of $\rho(G)$, and let H be the subgroup generated by S.) Since H is finite and acts irreducibly in V, V has an F-form W stable under the action of H. We claim that W is stable under the action of G. Thus let g be any element of G and let I be the subgroup of G generated by H and g. I is finite and acts irreducibly in V; and consequently I also stabilizes some F-form, say X, of V. The F-algebras A_H, A_I generated by $\rho(H)$, $\rho(I)$ respectively stabilize W, X respectively. By Burnside's Theorem [1, p. 182], $A_H|W = \mathrm{Hom}_F(W, W)$ and $A_I|X = \mathrm{Hom}_F(X, X)$. Thus $\dim_F A_H = \dim_F A_I = n^2$. Since $A_H \subseteq A_I$, we have $A_H = A_I$. Thus I stabilizes W. Thus g stabilizes W. Since g was chosen arbitrarily, G stabilizes W.

LEMMA.[2] *Let $S_1, S_2, \cdots$ be a sequence of finite nonempty sets. For each $i \geqq 2$, let f_i be a function from S_i into S_{i-1}. Then there exists a sequence $s_1, s_2, \cdots$ such that $s_i \in S_i$ and $f_i(s_i) = s_{i-1}$ for all $i \geqq 2$.*

PROOF. For convenience, let S_0 be a set consisting of a single element s_0 and let f_1 be the function from S_1 into S_0. Let $f_{ij} = f_i \circ f_{i+1} \circ \cdots \circ f_j$ for $i<j$. Suppose that a sequence $s_0, s_1, \cdots, s_n$ has been found such that for $i \leqq i \leqq n$ and $i<j$, $f_i(s_i) = s_{i-1}$ and $s_i \in f_{ij}(S_j)$. (If $n=0$, s_0 by itself is such a sequence). To prove the lemma, it suffices by induction to show that such a sequence can be augmented —that is, that there exists $s_{n+1} \in S_{n+1}$ such that $f_{n+1}(s_{n+1}) = s_n$ and $s_{n+1} \in f_{n+1,j}(S_j)$ for $n+1<j$. For this, choose for each $j>n+1$ an element x_j of S_j such that $f_{nj}(x_j) = s_n$; and for $j>n+1$, let $y_j = f_{n+1,j}(x_j)$. Then since S_{n+1} is finite, there exists y in S_{n+1} such that $y=y_j$ for infinitely many j. Now $y = s_{n+1}$ has the desired properties.

THEOREM 2. *G has a faithful completely reducible representation of degree n over K if and only if*

(1) *G is countable*; *and*

(2) *each finite subset of G is contained in a finite subgroup H which has a faithful completely reducible representation of degree n over K.*[3]

PROOF. Suppose first that G has a faithful completely reducible representation ρ in V over K. Then G is countable by Theorem 1. Let S be a finite subset of G. Let $V = \sum_1^m \oplus V_k$ where the V_k are irreducible G-subspaces of V. For each k, G has a finite subgroup H_k which

[2] This lemma is a special case of a theorem of König [3, Theorem 6, p. 81].

[3] Malcev, using quite different techniques, shows in [4] that a group G, every finitely generated subgroup of which has a faithful representation of degree n, has a faithful representation of degree n, though possibly over a much bigger field. In this general setting, one has no control over the ground field (as is illustrated by unipotent groups over large fields of nonzero characteristic).

is irreducible in V_k. (An argument for this is given in the proof of Theorem 1.) Let H be the subgroup generated by the set $S \cup H_1 \cup \cdots \cup H_m$. H is finite and contains S; and $(\rho | H, V)$ is a faithful completely reducible representation of H of degree n over K.

Now suppose that G satisfies the conditions (1) and (2). Let V be a vector space over K of dimension n. Then we can choose a chain $H_1 \subseteq H_2 \subseteq \cdots$ of finite subgroups H_i of G such that $G = \bigcup H_i$ and such that for each i, the set S_i of equivalence classes of faithful completely reducible representations of H_i in V over K is nonempty. The S_i are finite, and we proceed to define mappings $f_i: S_i \rightarrow S_{i-1}$ for $i \geqq 2$. For $i \geqq 2$, let ρ be a representative of an element of S_i. Pass from the representation $(\rho | H_{i-1}, V)$ to the direct sum (ρ', V') of its composition factors as representations of H_{i-1} over K. Then (ρ', V') is a completely reducible representation of H_{i-1} of degree n over K, and we claim that it is faithful. Thus let I be the kernel in H_{i-1} of (ρ', V'). $\rho(I)$ is then a normal unipotent subgroup of $\rho(H_{i-1})$. Since ρ is faithful, it follows that I is a p-group if K has characteristic $p > 0$ and that $I = \{1\}$ if K has characteristic 0. H_{i-1} has a faithful completely reducible representation ρ_{i-1} (since S_{i-1} is nonempty), and the preceding observation shows that $\rho_{i-1}(I)$ is a normal unipotent subgroup of $\rho_{i-1}(H_{i-1})$. As a normal subgroup of the completely reducible linear group $\rho_{i-1}(H_{i-1})$, $\rho_{i-1}(I)$ is completely reducible [1, p. 343]. And as a completely reducible linear unipotent group, $\rho_{i-1}(I) = \{1\}$ [2, pp. 775–776]. Since ρ_{i-1} is faithful, $I = \{1\}$. Thus (ρ', V') is faithful. Therefore (ρ', V') is equivalent to a representative of a unique element of S_{i-1}. Thus the mapping $(\rho, V) \rightarrow (\rho', V')$ induces a mapping $f_i: S_i \rightarrow S_{i-1}$. By the lemma, there exists a sequence $s_1, s_2, \cdots$ such that $s_i \in S_i$ and $f(s_i) = s_{i-1}$ for $i \geqq 2$. For each i, choose a representative ρ_i of s_i; and let $\boldsymbol{V}_i$ be the H_i-module over K defined by the representation (ρ_i, V). (The underlying vector space of $\boldsymbol{V}_i$ is V; and the action of H_i on $\boldsymbol{V}_i$ is given by ρ_i.) Then the construction of the f_i and the equations $f_i(s_i) = s_{i-1}$ show that for $i \geqq 2$, $\boldsymbol{V}_{i-1}$ and the direct sum $\boldsymbol{V}_i'$ of the composition factors of the restriction of $\boldsymbol{V}_i$ to H_{i-1} are equivalent as H_{i-1}-modules over K. For each i, choose a decomposition $\boldsymbol{V}_i = \sum_{k=1}^{n_i} \oplus \boldsymbol{V}_{i,k}$ of $\boldsymbol{V}_i$ where the $\boldsymbol{V}_{i,k}$ are irreducible H_i-submodules of $\boldsymbol{V}_i$ over K. Then $n_1 \geqq n_2 \geqq \cdots$. (If $i \geqq 2$, then $n_i \leqq n_{i-1}$; for by the equivalence of $\boldsymbol{V}_{i-1}$ and $\boldsymbol{V}_i'$, n_{i-1} is the number of composition factors of the restriction $\boldsymbol{V}_i | H_{i-1}$ of $\boldsymbol{V}_i$ to H_{i-1}.) For some j, we have $n_1 \geqq n_2 \geqq \cdots \geqq n_j = n_{j+1} = n_{j+2} = \cdots$. Thus for $i \geqq j+1$, $n_{i-1} = n_i$ and the series

$$\boldsymbol{V}_{i,1} \subset \boldsymbol{V}_{i,1} \oplus \boldsymbol{V}_{i,2} \subset \cdots \subset \sum_{k=1}^{n_i} \oplus \boldsymbol{V}_{i,k} = \boldsymbol{V}_i$$

of H_i-submodules of V_i is in fact a composition series for the restriction of V_i to H_{i-1}. (For if the above series were to admit an H_{i-1}-stable refinement, a composition series for the restriction of V_i to H_{i-1} would determine more than n_i composition factors, contradicting $n_{i-1}=n_i$.) It follows that for $i\geqq j+1$, the representation V_i' of H_{i-1} over K defined earlier is equivalent to the restriction $V_i|H_{i-1}$ of V_i to H_{i-1}. Thus V_{i-1} and $V_i|H_{i-1}$ are equivalent for $i\geqq j+1$. By suitably modifying the V_i successively up to equivalence, we may assume that $V_{i-1}=V_i|H_{i-1}$ for $i\geqq j+1$. The condition $V_{i-1}=V_i|H_{i-1}$ for $i\geqq j+1$ is that the underlying vector spaces of the V_i coincide for $i\geqq j+1$; and that if ρ_i: $H_i\rightarrow\mathrm{Hom}_K(V_i, V_i)$ denotes the action of H_i on the underlying vector space V_i of V_i for all i, then $\rho_s|H_r=\rho_r$ for $j+1\leqq r\leqq s$. Now if $\rho=\bigcup_{r\geqq j+1}\rho_r$ (that is, if ρ is the function with domain $G=\bigcup_{r\geqq j+1}H_r$ defined by $\rho(g)=\rho_r(g)$ if $g\in H_r$ and $r\geqq j+1$), then ρ is a faithful completely reducible representation of G of degree n.

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