ASYMPTOTIC BEHAVIOR OF MEROMORPHIC FUNCTIONS WITH EXTREMAL DEFICIENCIES

BY ALBERT EDREI AND ALLEN WEITSMAN[1]

Communicated by Maurice Heins, September 18, 1967

Let $f(z)$ be a meromorphic function; it is assumed that the reader is familiar with the following symbols of frequent use in Nevanlinna's theory

$$n(r,f), \quad N(r,f), \quad T(r,f), \quad \delta(\tau, f).$$

The lower order μ and the order λ of $f(z)$ are defined by the familiar relations

$$\liminf_{r\to\infty} \frac{\log T(r,f)}{\log r} = \mu, \qquad \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \lambda.$$

In addition to these classical concepts, we consider the *total deficiency* $\Delta(f)$ of the function f

$$\Delta(f) = \sum_{\tau} \delta(\tau, f)$$

where the summation is to be extended to all the values τ, finite or ∞, such that

$$\delta(\tau, f) > 0. \tag{1}$$

The number of deficient values of f, that is the number of distinct values of τ for which (1) holds, will be denoted by $\nu(f)$ ($\leqq +\infty$).

The investigation presented here leads to the proof of

THEOREM A. *Let $f(z)$ be a meromorphic function of lower order μ:*

$$\tfrac{1}{2} < \mu < 1, \tag{2}$$

and let the poles of $f(z)$ have maximum deficiency ($\delta(\infty, f) = 1$). Then

$$\Delta(f) \leqq 2 - \sin \pi\mu. \tag{3}$$

Moreover, if equality holds in (3), *then*

$$\nu(f) = 2. \tag{4}$$

---

[1] The authors gratefully acknowledge support by the National Science Foundation under grants GP-5019 and GP-7507.

Theorem A remains valid if, in (2) and (3), the lower order μ is replaced by the order λ. If we perform this substitution, the *assertion* $\lambda=\mu$ *follows from the assumption* $\Delta(f)=2-\sin\pi\lambda$.

With μ replaced by λ, the inequality (3) is known [4, Corollary 1.3, p. 235]; in its present form, which does not exclude functions of infinite order, it seems to be new. Concerning (4), we observe that one of us had already proved it for all μ belonging to the sequence $\{(1/2)+(1/2q)\}$ $(q=1, 2, 3, \cdots)$ [Notices Amer. Math. Soc., 1967, Abstract 643-23]. Theorem A remains true in the limiting case $\mu=1$; with the restriction[2] $\lambda<+\infty$ it follows from a sharpened form due to Edrei and Fuchs [3, Théorème 3, p. 264] of a result of Pfluger [6].

Finally we remark that, if (2) is replaced by $0\leqq\mu\leqq\frac{1}{2}$, *and if* $\delta(\infty, f)=1$, *then*

$$\Delta(f) = \delta(\infty, f) = 1$$

and $\nu(f)=1$. This follows immediately from an older result of one of us [1, Theorem 3, p. 4].

1. Auxiliary notions and notations. Our proof depends essentially on the following fact. If equality holds in (3), there must exist infinitely many, well chosen intervals

$$R_m' \leqq r \leqq R_m'' \qquad (m = 1, 2, 3, \cdots), \tag{1.1}$$

such that

$$\lim_{m\to\infty} R_m' = +\infty, \qquad \lim_{m\to\infty} \frac{R_m''}{R_m'} = +\infty,$$

and such that, if r and t lie in the intervals (1.1), then $T(t, f)/T(r, f)$ is very close to $(t/r)^\mu$.

A precise formulation requires a few definitions and notations:

I. Pólya peaks of order μ. *A positive sequence* $r_1, r_2, r_3, \cdots$ *of numbers tending to* $+\infty$ *is said to be a sequence of Pólya peaks, of order* μ *of* $T(r)$, *if it is possible to find three positive sequences* $\{r_m'\}$, $\{r_m''\}$, $\{\epsilon_m\}$, *such that, as* $m\to+\infty$,

$$r_m' \to +\infty, \quad (r_m/r_m') \to +\infty, \quad (r_m''/r_m) \to +\infty, \quad \epsilon_m \to 0,$$

and such that the inequalities

$$r_m' \leqq t \leqq r_m'' \qquad (m > m_0),$$

[2] Some of the arguments used in our proof of Theorem 1 would make it possible to omit this restriction.

imply

$$T(t)/T(r_m) \leqq (1+\epsilon_m)(t/r_m)^{\mu}.$$

We take for granted the fact that, if $f(z)$ is of lower order μ, then $T(r)=T(r, f)$ has a sequence of Pólya peaks of order μ. A proof will be found in [2, pp. 85–86].

II. QUANTITIES $u(f)$ AND $v(f)$ DEFINED IN TERMS OF EXCEPTIONAL SETS. Let $\mathcal{E}$ denote a measurable subset of the axis $r>0$ and let $\mathcal{E}[r', r'']$ be the portion of $\mathcal{E}$ which lies in the interval $[r', r'']$. We say that $\mathcal{E}$ has *density zero* if

$$\lim_{r\to\infty} \frac{\text{meas } \mathcal{E}[0, r]}{r} = 0.$$

We consider systematically the two quantities

$$(1.2) \qquad \limsup_{r\to\infty;\, r\notin\mathcal{E}} \frac{N(r, 1/f)}{T(r)} = u(f) = u, \qquad \limsup_{r\to\infty;\, r\notin\mathcal{E}} \frac{N(r,f)}{T(r)} = v(f) = v,$$

as well as the analogous quantities $u(f')$ and $v(f')$.

The set $\mathcal{E}$, which is avoided as $r\to+\infty$, *is always assumed to be of density zero.*

If $\mathcal{E}$ is a bounded set, the formulae (1.2) reduce to

$$u = 1 - \delta(0,f), \qquad v = 1 - \delta(\infty,f).$$

III. DEFINITIONS OF THE SECTOR $\mathcal{S}$ AND OF THE COUNTING FUNCTION $n(\mathcal{S}, f)$. The sector

$$\mathcal{S} = \mathcal{S}(\omega, \gamma; R', R'')$$

is defined to be the set of all points z satisfying the inequalities

$$\omega - \gamma \leqq \arg z \leqq \omega + \gamma \qquad (0<\gamma<\pi), \quad R' \leqq |z| \leqq R''.$$

We extend in an obvious way Nevanlinna's notation and denote by $n(\mathcal{S}, f)$ the number of poles of $f(z)$ which fall in the sector $\mathcal{S}$.

2. Statement of the main result.

THEOREM 1. *Let $f(z)$ be a meromorphic function of lower order μ $(0<\mu<1)$ and let u and v be defined by* (1.2).

I. *Then*

$$(2.1) \qquad \sin^2 \pi\mu \leqq u^2 + v^2 - 2uv\cos\pi\mu.$$

Moreover, $v \leqq \cos\pi\mu$ implies $u=1$ and $u\leqq\cos\pi\mu$ implies $v=1$.

II. *Let* $\{r_m\}$ *be a sequence of Pólya peaks of order* μ *of* $T(r)$ *and let* $E_\infty(r)$ *and* $E_0(r)$ *be sets of* θ $(-\pi \leqq \theta < \pi)$ *defined by*

$$E_\infty(r) = \{\theta: |f(re^{i\theta})| \geqq r^\alpha\}, \qquad E_0(r) = \{\theta: |f(re^{i\theta})| \leqq r^{-\alpha}\},$$

where α *is an arbitrary, nonnegative constant.*

Assume that equality holds in (2.1) *and that* $u<1$, $v<1$. *Then all the following limits exist and satisfy the relations stated*

$$\lim_{m\to\infty} \text{meas } E_\infty(r_m) = s(\infty) = \frac{2}{\mu}\cos^{-1} v \qquad \left(0 < \cos^{-1} v \leqq \frac{\pi}{2}\right),$$

$$\lim_{m\to\infty} \text{meas } E_0(r_m) = s(0) = \frac{2}{\mu}\cos^{-1} u \qquad \left(0 < \cos^{-1} u \leqq \frac{\pi}{2}\right),$$

$$s(0) + s(\infty) = 2\pi.$$

Moreover, there exist three positive sequences $\{R'_m\}$, $\{R''_m\}$, $\{\tilde{\epsilon}_m\}$ *such that, as* $m \to +\infty$,

$$R'_m \to +\infty, \quad r_m/R'_m \to +\infty, \quad R''_m/r_m \to +\infty, \quad \tilde{\epsilon}_m \to 0,$$

and such that

$$R'_m \leqq t \leqq R''_m \qquad (m > m_0),$$

imply

$$(2.2) \qquad (t/r_m)^\mu (1 + \tilde{\epsilon}_m)^{-1} \leqq T(t)/T(r_m) \leqq (t/r_m)^\mu (1 + \tilde{\epsilon}_m)$$

and

$$(2.3) \quad \mu u - \tilde{\epsilon}_m \leqq n(t, 1/f)/T(t) \leqq \mu u + \tilde{\epsilon}_m, \quad \mu v - \tilde{\epsilon}_m \leqq n(t, f)/T(t) \leqq \mu v + \tilde{\epsilon}_m.$$

There also exist a real sequence $\{\omega_m\}$ *and a positive sequence* $\{\eta_m\}$ *such that, as* $m \to +\infty$, $\eta_m \to 0$ *and*

$$(2.4) \qquad \begin{aligned} n(\mathcal{S}(\omega_m, \eta_m; R'_m, R''_m), 1/f) &= n(R''_m, 1/f) + o(T(r_m)), \\ n(\mathcal{S}(\omega_m + \pi, \eta_m; R'_m, R''_m), f) &= n(R''_m, f) + o(T(r_m)). \end{aligned}$$

Assertion II of the above theorem is closely related to a tauberian theorem of Edrei and Fuchs [**5**, Theorem 1, p. 340]. The inequalities (2.2) and (2.3) are "local." Their validity is confined to the intervals $[R'_m, R''_m]$ and examples show that the inequalities are no longer true for unrestricted values of t.

The relations (2.3) and (2.4) determine, in the annulus $R'_m \leqq |z| \leqq R''_m$, the moduli and the arguments of the zeros and poles of $f(z)$

with such precision that an asymptotic evaluation of $f(z)$ becomes possible on suitable circumferences.

The steps which lead to the proof of assertion (4) of Theorem A may be described as follows.

1. The simultaneous consideration of $f(z)$ and $f'(z)$ shows that $\Delta(f)=2-\sin \pi\mu$ $(\frac{1}{2}<\mu<1)$ and $\delta(\infty, f)=1$ imply

$$u(f') = \sin \pi\mu, \qquad v(f') = 0 . \tag{2.5}$$

2. The relations (2.5) make it possible to apply assertion II of Theorem 1 to $f'(z)$ and hence to obtain the asymptotic evaluation of $f'(z)$ on suitable circumferences. This shows that, on single arcs of these circumferences, f' is so small that f is practically constant. On the complementary arcs f is very large. It is easily shown that this behavior limits to two the number of deficient values of $f(z)$.

References

1. A. Edrei, *The deficiencies of meromorphic functions of finite lower order*, Duke Math. J. **31** (1964), 1–22.
2. ———, *Sums of deficiencies of meromorphic functions*, J. Analyse Math. **14** (1965), 79–107.
3. A. Edrei and W. H. J. Fuchs, *Valeurs déficientes et valeurs asymptotiques des fonctions méromorphes*, Comment. Math. Helv. **33** (1959), 258–295.
4. ———, *The deficiencies of meromorphic functions of order less than one*, Duke Math. J. **27** (1960), 233–250.
5. ———, *Tauberian theorems for a class of meromorphic functions with negative zeros and positive poles*, Proc. Int. Conf. in Function Theory, Erevan (1965), 339–358.
6. A. Pfluger, *Zur Defektrelation ganzer Funktionen endlicher Ordnung*, Comment. Math. Helv. **19** (1946/47), 91–104.

Syracuse University