THE POISSON-LAGUERRE TRANSFORM

BY DEBORAH TEPPER HAIMO AND FRANK M. CHOLEWINSKI[1]

Communicated by N. Levinson, September 21, 1967

For $\alpha \geqq 0$, let $L_n^\alpha(x)$ denote the Laguerre polynomial of degree n given by

$$L_n^\alpha(x) = (x^{-\alpha}e^{x}/n!)(d/dx)^n(x^{n+\alpha}e^{-x}), \qquad n = 0, 1, \cdots.$$

We define the Laguerre difference operator ∇_n by

$$\nabla_n f(n) = (n+1)f(n+1) - (2n+\alpha+1)f(n) + (n+\alpha)f(n-1).$$

Then the Laguerre difference heat equation is given by

$$(*) \qquad \nabla_n u(n, t) = \partial u(n, t)/\partial t.$$

A Laguerre temperature is a solution $u(n, t)$ of (*) which is a C^1 function of t. The fundamental Laguerre temperature is the function $g(n; t) = g(n, 0; t)$, where

$$g(n, m; t) = \int_0^\infty e^{-xt} L_n^\alpha(x) L_m^\alpha(x) d\Omega(x), \qquad t > 0,$$

with

$$d\Omega(x) = e^{-x}x^\alpha dx.$$

Corresponding to $g(n, m; t)$ is its conjugate $g(n^*, m; t)$ given by

$$g(n^*, m; t) = \int_0^\infty e^{-xt} L_n^\alpha(-x) L_m^\alpha(x) d\Omega(x), \qquad t > 0.$$

An important subclass of the class of Laguerre temperatures includes those Laguerre temperatures $u(n, t)$ which satisfy the condition

$$u(n,t) = \sum_{m=0}^\infty g(n, m; t-t')u(m, t')\rho(m), \qquad \rho(m) = m!/\Gamma(m+\alpha+1),$$

for every t, t', $0 < t' < t$, with the series converging absolutely. Laguerre temperatures which belong to this subclass are said to have the Huygens property. The functions $g(n, m; t)$ have this property.

[1] The research of the first author was supported in part by the National Aeronautics and Space Administration under Grant NGR-14-008-009 and that of the second author by the National Science Foundation under Grant GP-7167.

The Poisson-Laguerre transform of a function ϕ is formed with the kernel $g(n, m; t)$ and is given by

$$u(n, t) = \sum_{m=0}^{\infty} g(n, m; t)\phi(m)\rho(m),$$

whenever the series converges. Within its regions of convergence, the Poisson-Laguerre transform is a Laguerre temperature.

The object of this paper is to summarize the principal results derived in the development of an inversion and representation theory for the Poisson-Laguerre transform. Details and proofs will appear in [1].

Basic to the development of the theory are the properties of the functions $g(n, m; t)$, $g(n^*, m; t)$, and quotients of these. For example, the fact that, for $0<t<t_0\leqq 1$, $g(n, m; t)/g(n_0, m; t_0)$ is a positive monotone decreasing function of m for large m is a determining factor in the convergence behavior of the Poisson-Laguerre transform given in the following result.

THEOREM 1. *If $u(n, t)=\sum_{m=0}^{\infty} g(n, m; t)\phi(n)\rho(m)$ converges conditionally for (n_0, t_0), n_0 a nonnegative integer, $0<t_0\leqq 1$, then it converges for all $n=0, 1, \cdots$, and t, $0<t\leqq t_0\leqq 1$.*

Further, the boundedness of $g(n^*, m; t)/g(n^*, m; t_0)$ as a function of m enables us to establish the following convergence for the conjugate transform.

THEOREM 2. *If $u(n, t)=\sum_{m=0}^{\infty} g(n, m; t)\phi(m)\rho(m)$ converges absolutely for $t=t_0$, $0<t_0\leqq 1$, then so does*

$$u(n^*, t) = \sum_{m=0}^{\infty} g(n^*, m; t)\phi(m)\rho(m),$$

for $0<t\leqq t_0\leqq 1$.

Similar considerations are needed to develop the following fundamental inversion result.

THEOREM 3. *Let $\sum_{m=0}^{\infty} g(n, m; t)\phi(m)\rho(m)$ converge for (n_0, t_0), n_0 a nonnegative integer, $0<t_0\leqq 1$. Then*

$$\lim_{t\to 0^+} \sum_{m=0}^{\infty} g(n, m; t)\phi(m)\rho(m) = \phi(n).$$

We have, in addition, a conjugate inversion formula.

THEOREM 4. *Let* $u(n, t)=\sum_{m=0}^{\infty} g(n, m; t)\phi(m)\rho(m)$ *converge absolutely for* (n_0, t_0), n_0 *a nonnegative integer*, $0<t_0\leqq 1$. *Then*

$$\phi(n) = \sum_{m=0}^{\infty} g(n^*, m; t)u(m^*, t)\rho(m),$$

for $0<t\leqq t_0\leqq 1$.

The fact that a nonnegative Laguerre temperature has the Huygens property plays a central role in the characterization of Laguerre temperatures which can be represented by Poisson-Laguerre transforms of nonnegative functions. We have the following representation theorem.

THEOREM 5. *A necessary and sufficient condition that*

$$u(n, t) = \sum_{m=0}^{\infty} g(n, m; t)\phi(m)\rho(m)$$

with $\phi(m)$ *nonnegative and the series converging for* $n=0, 1, \cdots$, $0<t<c$ *is that* $u(n, t)$ *be a nonnegative Laguerre temperature there.*

REFERENCES

1. F. M. Cholewinski and D. T. Haimo, *Laguerre temperatures*, Proc. Conf. on New Directions in Orthogonal Expansions and their Continuous Analogues, Southern Illinois University Press, 1968.

2. I. I. Hirschman, Jr., *Laguerre transforms*, Duke Math. J. **30** (1963), 495–510.

SOUTHERN ILLINOIS UNIVERSITY, EDWARDSVILLE, AND
CLEMSON UNIVERSITY