DISCONJUGATE nth ORDER DIFFERENTIAL EQUATIONS AND PRINCIPAL SOLUTIONS[1]

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Consider the nth order linear differential equation

$$(0.1)\qquad P_n(D)[x] = 0, \qquad D[x] = dx/dt = x',$$

where $P_n(D) = P_n(t, D)$ and

$$(0.2)\qquad P_n(t, \lambda) = a_n(t)\lambda^n + \cdots + a_0(t), \quad a_n(t) > 0,$$

is a polynomial with real-valued, continuous coefficients on a t-interval I. §1 deals with disconjugacy criteria for (0.1). §2 deals with the existence of "principal" solutions for a disconjugate equation (0.1), as well as with the existence of solutions having specified estimates for their logarithmic derivatives. Proofs and related results will appear elsewhere.

1. Disconjugacy criteria. The differential equation (0.1) is said to be disconjugate (cf. [**9**], $n=2$) on I if no solution ($\not\equiv 0$) has n zeros, counting multiplicities, on I. If $u_1, \cdots, u_k$ are of class $C^{k-1}(I)$, we shall denote their Wronskian by $W(u_1, \cdots, u_k) = \det(D^{i-1}[u_j])$, for $i, j = 1, \cdots, k$. In particular, $W(u_1) \equiv u_1$.

DEFINITION. A set of functions $u_1, \cdots, u_{n-1}$ of class $C^n(I)$ is said to have property W (Pólya [**7**]) or to be a $w_n(I)$-system if

$$(1.1)\qquad W(u_1, \cdots, u_k) > 0 \quad \text{for } k = 1, \cdots, n-1.$$

DEFINITION. A set of functions $u_1, \cdots, u_{n-1}$ of class $C^n(I)$ is said to be a $W_n(I)$-system if, for $k = 1, \cdots, n-1$ and all sets of indices $(1 \leqq) i(1) < \cdots < i(k) (\leqq n-1)$,

$$(1.2)\qquad W(u_{i(1)}, \cdots, u_{i(k)}) > 0 \text{ on } I,$$

or, equivalently,

$$W(u_j, u_{j+1}, \cdots, u_k) > 0 \quad \text{for } 1 \leqq j \leqq k \leqq n-1.$$

In particular, (1.2) implies that $u_k > 0$ and that

$$(1.3)\qquad u_1'/u_1 < \cdots < u_{n-1}'/u_{n-1}.$$

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Since (0.1) is disconjugate on I if and only if it is disconjugate on every compact subinterval, it suffices to give disconjugacy criteria for the case that I is compact.

THEOREM 1.1. (i) *A sufficient condition for* (1.1) *to be disconjugate on a compact interval I is that there exist a $W_n(I)$-system of functions $u_1, \cdots, u_{n-1}$ satisfying*

$$(1.4) \qquad (-1)^{n-k}P_n(D)[u_k] \geqq 0 \quad \text{for } k = 1, \cdots, n-1.$$

(ii) *A necessary condition is that there exist a $W_n(I)$-system of solutions.*

Part (i) of this result can be considered a generalization of Sturm's comparison theorem for $n=2$ in a form used by Bôcher, de la Vallée Poussin, and Wintner (cf., e.g., [2, Theorem 7.2, p. 362]). Part (ii) is related to a result of Pólya [7] (cf. [2, Exercise 8.3, p. 67 and p. 560]), which states that if (0.1) is disconjugate on a half-open interval I and if I^0 is the interior of I, then (0.1) has a $w_n(I^0)$-system of solutions. In this direction, results of §2 imply

PROPOSITION 1.1. *The differential equation* (0.1) *is disconjugate on an open or half-open interval I if and only if it possesses a $w_n(I^0)$-system of solutions.*

The "only if" portion is false if w_n is replaced by W_n. From Theorem 1.1, we can deduce (with $u_k = \exp \alpha_k t$)

COROLLARY 1.1. *Assume that the polynomial $P_n(t, \lambda)$ has only real zeros $\lambda_1(t) \leqq \cdots \leqq \lambda_n(t)$ separated by constants $\alpha_1, \cdots, \alpha_{n-1}$, that is, $\lambda_1(t) \leqq \alpha_1 \leqq \lambda_2(t) \leqq \cdots \leqq \alpha_{n-1} \leqq \lambda_n(t)$. Then* (0.1) *is disconjugate on I.*

This sharpens one of the results of [3] obtained by very different methods. The proof of Theorem 1.1 and of most of the other theorems described here is by an induction on n. The success of this method depends on two factors: (1) the notion of a W_n-system which goes into a W_{n-1}-system during the induction and which together with (1.4) assures that $(-1)^{n-k+1}\beta_k \geqq 0$ for $k=0, \cdots, n-2$ in (1.6); (2) the reduction from n to $n-1$ by using the existence of a positive solution of (0.1) with suitable "monotony" properties.

The existence statement in (2) is obtained by transforming (0.1) into a suitable first order system making the results of Hartman and Wintner [4] (cf. [2, pp. 506–510]) available. If $u_1, \cdots, u_{n-1}$ is a $w_n(I)$-system and $u_n \in C^n(I)$ satisfies $W(u_1, \cdots, u_n) > 0$, let $\omega_k = W(u_1, \cdots, u_k)$ for $k=1, \cdots, n$ and $\omega_0 = 1$. We can write

$$P_n(D)[x] = \sum_{i=0}^{n} \beta_i(t) W(u_1, \cdots, u_i, x), \quad \beta_n(t) > 0,$$

where $\beta_0, \cdots, \beta_n$ are uniquely determined, continuous functions (and $W(u_1, \cdots, u_i, x) = x$ if $i = 0$). Correspondingly, if the vector $y = (y_1, \cdots, y_n)$ is defined by

$$y_j = W(x, u_1, \cdots, u_{j-1})/\omega_j \quad \text{for } j = 1, \cdots, n, \tag{1.5}$$

then (0.1) is equivalent to the first order system for y,

$$\begin{aligned} y_j' &= -(\omega_j^2/\omega_{j-1}\omega_{j+1}) y_{j+1} \quad \text{for } j = 1, \cdots, n-1, \\ y_n' &= -(\omega_{n-1}/\beta_n \omega_n^2) \sum_{k=1}^{n} (-1)^{n-k} \beta_{k-1} \omega_k y_k. \end{aligned} \tag{1.6}$$

We can choose u_n so that $\beta_{n-1} \equiv 0$. Also, if $u_1, \cdots, u_{n-1}$ is a $W_n(I)$-system satisfying (1.4), then we can show that $(-1)^{n-k}\beta_{k-1}(t) \geqq 0$ for $k = 1, \cdots, n-1$.

2. **Principal solutions.** The next result serves to define and give some of the properties of 1-*st*, 2-*nd*, $\cdots$, $(n-1)$-*st principal solutions* $\xi_1(t), \cdots, \xi_{n-1}(t)$ (at $t = \beta$) of an equation (0.1) disconjugate on (α, β).

THEOREM 2.1. *Let* (0.1) *be disconjugate on an open interval* $I = (\alpha, \beta)$, $-\infty \leqq \alpha < \beta \leqq \infty$. *Then there exists a set of solutions* $\xi_1, \cdots, \xi_{n-1}$ *with the following properties*

(i) $\xi_1 > 0$ *on* I *and is unique up to positive constant factors*; *for* $k = 2, \cdots, n-1$, $\xi_k > 0$ *for* t *near* β *and is unique up to positive constant factors and addition of linear combinations of* $\xi_1, \cdots, \xi_{k-1}$.

(ii) $\xi_1, \cdots, \xi_{n-1}$ *is a* $w_n(I)$-*system.*

(iii) *For* $j = 1, \cdots, n-2$, $\xi_j/\xi_{j+1} \to 0$ *as* $t \to \beta$. *If* $x(t)$ *is a solution of* (0.1) *linearly independent of* $\xi_1, \cdots, \xi_k$, *then* $\xi_k/x \to 0$ *as* $t \to \beta$.

(iv) *If* $\alpha < \gamma < \beta$, $I_\gamma = (\gamma, \beta)$, *and* $u_1, \cdots, u_{n-1}$ *is a* $w_n(I_\gamma)$-*system of solutions or a* $W_n(I_\gamma)$-*system satisfying* (1.4), *then, on* I_γ,

$$\xi_1'/\xi_1 \leqq u_1'/u_1 \quad \text{and} \quad W(\xi_1, u_1, \cdots, u_k) \geqq 0 \quad \text{for } k = 1, \cdots, n-1.$$

In particular, for $\gamma < t < \beta$, $\xi_1'(t)/\xi_1(t) = \inf x_1'(t)/x_1(t)$, *where the infimum is taken over* $\{x_1$: *there exists a* $w_n(I_\gamma)$-*system of solutions* $x_1, \cdots, x_{n-1}\}$.

(v) *If* $x = \xi(t, \gamma)$ *is the solution of* (0.1) *satisfying*

$$x = D[x] = \cdots = D^{n-2}[x] = 0, \ (-1)^{n-1}D^{n-1}[x] > 0 \text{ at } t = \gamma,$$

$$\sum_{i=0}^{n-1} | D^i[x] |^2 = 1 \text{ at a point, independent of } \gamma,$$

then $\xi_1(t) = C^n(I) - \lim \xi(t, \gamma)$ *as* $\gamma \to \beta$.

Properties analogous to (iv) and (v) for $\xi_2, \cdots, \xi_{n-1}$ are obtained, in addition to characterizations of $\xi_2, \cdots, \xi_{n-1}$ under transformations of (0.1) (e.g., under the variation of constants $x = \xi_1 v$).

The idea of a principal solution in the case $n=2$ originated with Leighton and Morse [5] (cf. [2, pp. 350–361]).

THEOREM 2.2. *Let there exist a* $W_n(I)$*-system* $u_1, \cdots, u_{n-1}$ *satisfying* (1.4). *Then* (0.1) *has positive, linearly independent solutions* $x_1, \cdots, x_n$ *satisfying*

$$(2.1) \qquad x_1'/x_1 \leqq u_1'/u_1 \leqq x_2'/x_2 \leqq \cdots \leqq u_{n-1}'/u_{n-1} \leqq x_n'/x_n$$

on I. If, in addition, there is a function u_0 [*and/or* u_n] *of class* $C^n(I)$ *satisfying* $(-1)^n P_n(D)[u_0] \geqq 0$ [*and/or* $P_n(D)[u_n] \geqq 0$] *and, for* $k = 1, \cdots, n-1$,

$$u_0 > 0, \quad W(u_0, \cdots, u_k) \geqq 0 \quad [\text{and/or } u_n > 0, W(u_k, \cdots, u_n) \geqq 0],$$

then x_1 [*and/or* x_n] *can be chosen to satisfy*

$$(2.2) \quad u_0'/u_0 \leqq x_1'/x_1 \leqq u_1'/u_1 \; [\text{and/or } u_{n-1}'/u_{n-1} \leqq x_n'/x_n \leqq u_n'/u_n].$$

If $n=2$, the result concerning (2.1) is essentially a theorem of A. Kneser (cf. [2, Corollary 6.4, p. 357]) applied after the variation of constants $x = u_1 v$. Under different conditions on u_0, u_1 and u_2, Olech [6] obtained Theorem 2.2. for $n=2$.

COROLLARY 2.1. *If* $\alpha_1 < \cdots < \alpha_{n-1}$ *in Corollary* 1.1, *then* (0.1) *has positive, linearly independent solutions* $x_1, \cdots, x_n$ *satisfying*

$$x_1'/x_1 \leqq \alpha_1 \leqq x_2'/x_2 \leqq \cdots \leqq \alpha_{n-1} \leqq x_n'/x_n$$

on I. If, in addition, there is a number α_0 [*and/or* α_n] *satisfying* $\alpha_0 \leqq \lambda_1(t)$ [*and/or* $\lambda_n(t) \leqq \alpha_n$], *then* x_1 [*and/or* x_n] *can be chosen to satisfy*

$$\alpha_0 \leqq x_1'/x_1 \leqq \alpha_1 \; [\text{and/or } \alpha_{n-1} \leqq x_n'/x_n \leqq \alpha_n].$$

This result is given for $n=2$ by Olech [6]. For $n=3$, Schuur [8] obtained the existence of x_2 (but not x_1, x_3) for $I = [0, \infty)$. In his talk [8], Schuur mentioned that another proof for the existence of x_2 was given by L. Jackson (in a paper not available to me) by considering the second order, nonlinear differential equation for $r = x'/x$. In this form, Schuur's result is contained in Hartman [1] (cf. [2, Theorem 5.2, p. 434]) after the translation of the variable $r \to r-(\alpha_1+\alpha_2)/2$.

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