A TOPOLOGICAL CLASSIFICATION OF CERTAIN 3-MANIFOLDS

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In this paper we shall prove that a compact 3-manifold, which can be fibered over S^1 is topologically determined by its fundamental group and the subgroups belonging to its boundary components.

This theorem was first supposed by J. Stallings [2] and proved in the case of a closed manifold by L. Neuwirth [1]. His proof for bounded orientable manifolds is not complete.

Let M be such a compact 3-manifold with boundary components $B_1, B_2, \cdots, B_r, r \geqq 0$. We denote by $[A_i]$ the class of conjugated subgroups in $\pi_1(M) = G$ generated by loops on B_i. We call $\{G, [A_1], \cdots, [A_r]\}$ the peripheral system of M.

A fibering of M over S^1 is obtained from the product $F \times I$ of a compact surface F and the unit interval I by identifying $(F \times 0)$ and $(F \times 1)$ by a homeomorphism ζ of F. The boundary components of M are tori and Klein-bottles. We write $M = F \times I/\zeta$.

ζ and ζ^* belong to the same class $[\zeta]$ of homeomorphisms, if they are connected by the following operations:

(i) isotopic deformation

(ii) conjugation with a homeomorphism of F

(iii) replacing ζ by ζ^{-1}.

$F \times I/\zeta$ and $F \times I/\zeta^*$ are homeomorphic if ζ and ζ^* belong to the same class. The proof is immediate (see [1]). We consider the fundamental group N of F as a subgroup of G. The projection $M \to S^1$ induces a homomorphism χ: $G \to Z$ with kernel N. $\chi(A_i)$, $A_i \in [A_i]$, is a subgroup of Z with finite index n_i. n_i is the number of boundary curves of F contained in one boundary component B_i of M.[1] F has $n = \sum_{i=1}^{r} n_i$ boundary curves. If $n = 0$, then by N the type of F is given. For $n > 0$, N is a free group, and we have to decide whether F is orientable or not. Let c_{i0} be some boundary curve of F belonging to B_i, and choose $A_i \in [A_i]$ such that the generator of the infinite cyclic group $A_i \cap N$ is represented by c_{i0} resp. its inverse c_{i0}^{-1}. (The trivial case where F is a disk is excluded.) The cyclic groups $t^j A_i t^{-j} \subset N$, $j = 1, 2, \cdots, n_i - 1$ are generated by the boundary curves $c_{i,j}^{\epsilon_{ij}}$, $j = 1, \cdots, n_i - 1$, $\epsilon_{ij} = \pm 1$, of F, contained in B_i. The loops c_{ij}, $i = 1, \cdots, r$, $j = 0, 1, \cdots, n_i - 1, \cdots, \epsilon_{ij} = \pm 1$, are as elements of N determined modulo conjugation in N.

[1] The case $n_i > 1$ was not observed in [1].

LEMMA. *F is orientable if and only if for some choice of the exponents* $\epsilon_{ij} = \pm 1$

$$\prod_{i=1}^{r} \prod_{j=0}^{n_i - 1} c_{ij}^{\epsilon_{ij}} \in [N, N].$$

($[N, N]$ denotes the commutator subgroup of N.) The proof of the lemma is postponed.

F (assuming boundary curves) may now be wholly described in algebraic terms: $n = \sum_{i=1}^{r} n_i$ is the number of boundary curves of F. The lemma is a tool to decide about orientability. Knowing this, the rank of N gives the genus of F.

We now set out to describe ζ. ζ induces an automorphism α of N. The automorphism class of α (modulo inner automorphisms) can be represented by

$$\alpha : a \to tat^{-1}, \qquad a \in N.$$

Replacing t by t^{-1} means replacing α by α^{-1}.

THEOREM. *Let* $M = F \times I / \zeta$ *be a fibered 3-manifold with peripheral system* $\{G, [A_i]\}$, *and normal subgroups* N, $\chi: G \to Z$, $\chi^{-1}(0) = N$. *Suppose, that* M^* *is another irreducible manifold with peripheral system* $\{G^*, [A_2^*]\}$, *and* Φ *an isomorphism between* G *and* G^* *mapping* $[A_i]$ *onto* $[A_i^*]$. *Then* M *and* M^* *are homeomorphic.*

PROOF. Define $N^* = \Phi(N)$, and $\chi^*: G \to Z^*$, $\chi^{*-1}(0) = N^*$. Then $M^* = F^* \times I / \zeta^*$ by Stallings theorem [2]. As the systems of groups $\{G, N, [A_i]\}$ and $\{G^*, N^*, [A_i^*]\}$ are isomorphic we can deduce

$$n_i^* = [Z^* : \chi^*(A_i^*)] = [Z : \chi(A_i)] = n_i.$$

By the argument given above it follows that F and F^* are homeomorphic, ζ^* induces the automorphism class $[\alpha^*]$,

$$\alpha^* : a^* \to t^* a^* t^{*-1}, \qquad a^* \in N^*,$$

$$\text{or } [\alpha^{*-1}], \text{ i.e. } [\phi \alpha \phi^{-1}] = [\alpha^*] \quad \text{or} \quad [\phi \alpha \phi^{-1}] = [\alpha^{*-1}], \quad \phi = \Phi \mid N.$$

As Φ maps the peripheral system of M onto that of M, ϕ maps the peripheral system of F onto that of F^*. We may therefore apply the Nielsen theorem for bounded surfaces [3]. It follows, that ϕ is induced by a homeomorphism $\eta: F \to F^*$.

By the Baer theorem [3] $\eta \zeta \eta^{-1}$ and ζ^* resp. ζ^{*-1} are isotopic, hence they belong to the same class $[\zeta^*]$. This implies $F^* \times I / \zeta^*$ is homeomorphic $F \times I / \zeta$. It remains to prove the lemma: Any compact non-orientable surface possesses a system of canonical curves $a_1, \cdots, a_n$,

$b_1, \cdots, b_m$ with $\prod_i a_i \prod_j b_j^2$ homotopic to zero. We maintain that this can be achieved for any chosen orientation of the boundary curves r_i, where $s_i r_i s_i^{-1}$ represents a_i.

As we can permute the a_i by braid automorphisms, it suffices to show, that r_n can be replaced by r_n^{-1}: $\cdots a_n b_1^2 \cdots = \cdots a_n b_1 a_n^{-1} \cdot a_n b_1 \cdots = b_1' a_n^{-1} b_1'^{-1} b_1'^2 \cdots = \cdots a_n' b_1'^2 \cdots$ putting $a_n b_1 = b_1'$, $a_n' = b_1' a_n^{-1} b_1'^{-1}$. Obviously a_n' represents $s_n' r_n^{-1} s_n'^{-1}$.

But it is easily seen by abelianizing that $\prod_i a_i = (\prod_j b_j^2)^{-1} \in [N, N]$. The other part of the lemma is trivial.

References

1. L. Neuwirth, *A topological classification of certain* 3 *manifolds*, Bull. Amer. Math. Soc. 69 (1963), 372–375.

2. J. Stallings, "On fibering certain 3-manifolds," in *Topology of 3-manifolds and related topics*, Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 95–100.

3. H. Zieschang, *Über Automorphismen ebener diskontinuierlicher Gruppen*, Math. Ann. 166 (1966), 148–167.

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