ON THE COMMENSURABILITY CLASS OF THE SIEGEL MODULAR GROUP

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The purpose of this note is to determine the commensurability class of $G_{\mathfrak{o}}$, where G is either the Sympletic Group Sp_n or the Special Linear Group Sl_n, and $\mathfrak{o}$ is the ring of integers of a number field k of finite degree over the rationals. This problem has been solved in the local case by Hijikata [6], and Bruhat [3], for Sl_n and by Allan [1] and Bruhat-Tits [8] for Sp_n. The only known global solution is due to Helling [5], in the case of Sl_2. We shall exhibit a countable family of arithmetic groups in G, such that every maximal arithmetic group is conjugate to one group of the family, and if $\mathfrak{o}$ is a principal ideal ring, every group of this family is a maximal arithmetic group.

We know that if G is a connected semisimple linear group defined over k, which is absolutely irreducible as a matrix group, then a maximal arithmetic group Δ is the normalizer of its intersection Δ' with G_k, and Δ' is selfnormalizer in G_k. Our problem is then to determine the selfnormalizer subgroups of G_k whose normalizer is maximal, because the maximal groups which are the normalizer of maximal subgroups of G_k has already been determined in [1], for the groups we are interested in.

Let $\mathfrak{p}$ be a finite prime spot of k and $k(\mathfrak{p})$ be the completion of k at $\mathfrak{p}$; let $\mathfrak{o}(\mathfrak{p})$ be the ring of integers of $k(\mathfrak{p})$. Let Δ be an arithmetic group contained in G_k and let $\Delta^{\mathfrak{p}}$ be its $\mathfrak{p}$-adic closure in $G_{k(\mathfrak{p})}$. Assume that $N_k(\Delta)=\Delta$, and let $\mathfrak{J}$ be the ideal in $\mathfrak{o}$ generated by all $\mu\in\mathfrak{o}$ such that μg has only algebraic integral entries for all $g\in N(\Delta)$. Let $\Omega(\mathfrak{p})$ be the algebraic closure of $k(\mathfrak{p})$. It is easy to see that $N(\Delta)$ can be imbedded in $N(\Delta^{\mathfrak{p}})$. Clearly the groups Δ which we want to determine are such that $\Delta=\bigcap(G_k\cap\Delta^{\mathfrak{p}})$, the intersection taken over all finite primes of k. From now on we shall assume that all groups Δ considered, have this property. We shall also assume that G has the strong approximation property (see [7]). Then it is very simple to verify the following two lemmas:

LEMMA 1. *$N_k(\Delta)=\Delta$ if and only if $N_{k(\mathfrak{p})}(\Delta^{\mathfrak{p}})=\Delta^{\mathfrak{p}}$ for all $\mathfrak{p}$. Similarly Δ is maximal in G_k if and only if $\Delta^{\mathfrak{p}}$ is maximal in $G_{k(\mathfrak{p})}$ for all $\mathfrak{p}$.*

LEMMA 2. *Δ is conjugate to Γ in G_k if and only if $(\Delta)^{\mathfrak{p}}$ is conjugate to $(\Gamma)^{\mathfrak{p}}$ in $G_{k(\mathfrak{p})}$ for all $\mathfrak{p}$.*

For each $\mathfrak{p}$, let $\Sigma_{\mathfrak{p}}$ be a finite set of open compact subgroups of $G_{k(\mathfrak{p})}$ such that

1. Every Δ in $\Sigma_{\mathfrak{p}}$ is selfnormalizer in $G_{k(\mathfrak{p})}$.
2. $N(\Delta)$ is maximal in $G_{\Omega(\mathfrak{p})}$, for all $\Delta \in \Sigma_{\mathfrak{p}}$.
3. If Δ' is a maximal arithmetic group in $G_{\Omega(\mathfrak{p})}$, then $\Delta' = N(\Delta)$, where Δ is conjugate in $G_{k(\mathfrak{p})}$ to some group in $\Sigma_{\mathfrak{p}}$.

Let Σ be the set of all arithmetic groups in G_k, such that their $\mathfrak{p}$-adic completion lies in $\Sigma_{\mathfrak{p}}$ for all $\mathfrak{p}$.

THEOREM 1. *If Δ' is maximal arithmetic group in G, and $\Delta = \Delta' \cap G_k$, then $\Delta' = N(\Delta)$, and Δ is conjugate in G_k to an arithmetic group in Σ.*

The proof of this theorem is an immediate consequence of the above lemmas.

2. *Case* $G = \mathrm{Sl}_n(C)$. Let $\mathfrak{A}_{ij}$, $i, j = 1, \cdots, n$ be n^2 fractional ideals in k such that $\mathfrak{A}_{ij}\mathfrak{A}_{jk} = \mathfrak{A}_{ik}$, for all $i, j, k = 1, \cdots, n$, and $\mathfrak{A}_{ij}$ is nth power free. Let $\mathfrak{B}_{ij}$ be integral ideals such that

1. $\mathfrak{B}_{ij} = \mathfrak{o}$ if $i \geqq j$ and $\mathfrak{B}_{ij}\mathfrak{B}_{jk} = \mathfrak{B}_{ik}$, for all $i, j, k = 1, \cdots, n$.
2. $\mathfrak{B}_{ij} \mid \mathfrak{B}_{im}$ and $\mathfrak{B}_{tj}$ for all $t \leqq i$ and $m \geqq j$, and $\mathfrak{B}_{ij}$ is square free for all $i, j = 1, \cdots n$.
3. For each $\mathfrak{p} \mid \mathfrak{B}_{1n}$ there exists $\mathfrak{d} \mid \mathfrak{B}_{1n}$ such that if $s = s(\mathfrak{d})$ is defined by $\mathfrak{d} \mid \mathfrak{B}_{1s+1}$ but $\mathfrak{d} \nmid \mathfrak{B}_{1s}$, then $s \mid n$, $s(\mathfrak{p}) = s(\mathfrak{d})$, and if $n = s(\mathfrak{d})n(\mathfrak{d})$, then the ideal class of $\mathfrak{d}$ is an $n(\mathfrak{d})$th power of a some ideal class in k.
4. For all divisors $\mathfrak{d}$ of $\mathfrak{B}_{1n}$ and all $m = 1, \cdots, n(\mathfrak{d})$, $\mathfrak{d} \mid \mathfrak{B}_{ij}$ for all $i \leqq ms(d)$ and for all $j > ms(d)$.

Let $L(\mathfrak{A}, \mathfrak{B})$ be the direct summand order $L = (L_{ij})$ where $L_{ij} = \mathfrak{A}_{ij}\mathfrak{B}_{ij}$. We remark that $L(\mathfrak{o}, \mathfrak{B})^{\mathfrak{p}}$ the order obtained by Hijikata [5], and every order $L(\mathfrak{A}, \mathfrak{B})^{\mathfrak{p}}$ is conjugate to one of such orders in $\mathrm{Sl}_n(\Omega(\mathfrak{p}))$. Let $\Delta(\mathfrak{A}, \mathfrak{B}) = L(\mathfrak{A}, \mathfrak{B}) \cap \mathrm{Sl}_n(k)$. Let Σ' the family of all $\Delta(\mathfrak{A}, \mathfrak{B})$, and let $\Sigma_{\mathfrak{p}}$ be the set of all $\Delta(\mathfrak{A}, \mathfrak{B})^{\mathfrak{p}}$. We observe that the class Σ consists of all $\Delta(\mathfrak{A}, \mathfrak{B})$, where we drop the condition 3 for $\mathfrak{B}_{ij}$.

THEOREM 2. *If $N_k(\Delta) = \Delta$ and $N(\Delta)$ is maximal in G, then Δ is conugate in $\mathrm{Sl}_n(k)$ to some $\Delta(\mathfrak{A}, \mathfrak{B}) \in \Sigma'$.*

2. *$\Delta \in \Sigma'$ then $N(\Delta)$ is maximal and $N(\Delta)/\Delta \simeq \mathfrak{U}_n \times \mathfrak{C}_n(\mathfrak{B}_{1n}) \times \mathfrak{g}_n(\mathfrak{B}_{1n})$ (See [1], description of $N(\Delta)/\Delta$).*

The proof is obtained from Theorem 1 and by showing that for each $\mathfrak{d} \mid \mathfrak{B}_{1n}$ we can construct a matrix $g \in N(\Delta)$, $g = (g_{ij})$, such that $(g_{ij})^{n(\mathfrak{d})} = (\mathfrak{A}'_{ij})^{n(\mathfrak{d})}/\mathfrak{d}$, where $C(\mathfrak{A}_{ij})^{n(\mathfrak{d})} = C(\mathfrak{d})$. The proof of the maximality follows from the description of $N(\Delta)/\Delta$.

2. *Case* $G = \mathrm{Sp}_n(C)$. Let $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ and $\mathfrak{B}_1, \cdots, \mathfrak{B}_n$ be integral ideals such that

1. $\mathfrak{A}_i \mid \mathfrak{A}_j$ and $\mathfrak{B}_i \mid \mathfrak{B}_j$, for all $1 \leqq i \leqq j \leqq n$.

2. $\mathfrak{A}_n/\mathfrak{A}_1$ and $\mathfrak{B}_n/\mathfrak{B}_1$ are square free and $(\mathfrak{B}_n, \mathfrak{A}_n/\mathfrak{A}_1)=1$. We shall assume that when $n=2s$ is even, then $\mathfrak{B}_s=\mathfrak{B}_{s+1}$.

3. If $\mathfrak{q}$ is a nonprincipal prime, with $\mathfrak{q}\mid\mathfrak{B}_n$, then there exists $\mathfrak{d}\mid\mathfrak{B}_n$ such that $\mathfrak{q}\mid\mathfrak{d}$ and the ideal class of $\mathfrak{d}$ is a square.

We set

$$\begin{aligned}\mathfrak{L} &= \mathfrak{o}e_1 + \cdots + \mathfrak{o}e_n + \mathfrak{A}_1\mathfrak{B}_1e_{n+1} + \cdots + \mathfrak{A}_n\mathfrak{B}_ne_{2n} \text{ and}\\ \mathfrak{L}' &= \mathfrak{o}e_1 + \cdots + \mathfrak{o}e_n + \mathfrak{A}_1\mathfrak{B}_1' e_{n+1} + \cdots + \mathfrak{A}_n\mathfrak{B}_n' e_{2n}\end{aligned} \tag{1}$$

where $\mathfrak{B}_j'=(\mathfrak{B}_n/\mathfrak{B}_{n+1-j})$. If $\mathfrak{d}\mid\mathfrak{B}_n/\mathfrak{B}_1$, we set $s=s(\mathfrak{d})$, the index s such that $\mathfrak{d}\nmid\mathfrak{B}_s/\mathfrak{B}_1$ but $\mathfrak{d}\mid\mathfrak{B}_{s+1}/\mathfrak{B}_1$. If n is odd, we set $\mathfrak{J}'=\mathfrak{o}$ and if n is even, say $n=2s$, then we set $\mathfrak{J}'$ be the product of all primes $\mathfrak{p}$ dividing $\mathfrak{A}_{s+1}/\mathfrak{A}_s$ such that there exists a divisor $\mathfrak{d}$ of this ideal such that $\mathfrak{p}\mid\mathfrak{d}$ and the ideal class of $\mathfrak{d}$ is a square. We shall denote by $\Delta(\mathfrak{A}, \mathfrak{B})$ the subgroup of $\mathrm{Sp}_n(k)$ consisting of all matrices $g\in\mathrm{Sp}_n(k)$ such that $g\mathfrak{L}=\mathfrak{L}$ and $g\mathfrak{L}'=\mathfrak{L}'$.

THEOREM 3. *If Δ is an arithmetic group in $\mathrm{Sp}_n(k)$ such that $N_k(\Delta)=\Delta$ and $N(\Delta)$ is maximal, then Δ is conjugate in $\mathrm{Sp}_n(k)$ to some $\Delta(\mathfrak{A}, \mathfrak{B})$.*

2. *If for all primes $\mathfrak{p}$ dividing $\mathfrak{B}_n$ such that it satisfies 3, we have $s(\mathfrak{p})=s(\mathfrak{d})$, then $N(\Delta(\mathfrak{A}, \mathfrak{B}))$ is maximal, and $N(\Delta(\mathfrak{A}, \mathfrak{B}))/\Delta(\mathfrak{A}, \mathfrak{B}) \simeq \mathfrak{U}_2\times \mathfrak{C}_2(\mathfrak{J})\times\mathfrak{G}_2(\mathfrak{J})$, where $\mathfrak{J}=\mathfrak{J}'\,\mathfrak{B}_n$.*

We shall sketch the proof. Let $\Sigma_{\mathfrak{p}}$ be the family consisting of all $\Delta(\mathfrak{A}, \mathfrak{B})^{\mathfrak{p}}$, hence Σ is the family of all $\Delta(\mathfrak{A}, \mathfrak{B})$ obtained by dropping the condition 3 on $\mathfrak{B}_n$. We denote by Σ' be the family of all $\Delta(\mathfrak{A}, \mathfrak{B})$ and Σ'' be the subfamily consisting of those groups satisfying the hypothesis of the theorem. Our first assertion follows from Theorem 1, and from the fact that if $\Delta(\mathfrak{A}, \mathfrak{B})\in\Sigma$, then $N(\Delta(\mathfrak{A}, \mathfrak{B}))$ is maximal if and only if for all $\mathfrak{p}\mid\mathfrak{B}_n$ such that there is no $g\in N(\Delta(\mathfrak{A}, \mathfrak{B}))$, $g=(g_{ij})$ and $2\nmid \mathrm{ord}_{\mathfrak{p}}(g_{ij})^2$, then $\Delta(\mathfrak{A}, \mathfrak{B})^{\mathfrak{p}}$ is maximal in $\mathrm{Sp}_n(k(\mathfrak{p}))$. To get our second assertion for each $\mathfrak{d}\mid\mathfrak{J}$ such that the class of $\mathfrak{d}$ is a square, we construct an element g in $N(\Delta(\mathfrak{A}, \mathfrak{B}))$ such that $g=(g_{ij})$, $((g_{ij})^2) = (\mathfrak{A}_{ij}')^2/\mathfrak{d}$ for all $i, j=1, \cdots, 2n$, where $\mathfrak{A}_{ij}$ are integral ideals.

We would like to point out that if k has class number one, then $\Sigma=\Sigma'=\Sigma''$. Also our family is not the smallest as possible, i.e., there are pairs of $\Delta\in\Sigma'$ which are conjugate in $\mathrm{Sp}_n(\boldsymbol{C})$. Hence,

COROLLARY. *If Γ is the Siegel Modular Group, then up to conjugacy in $\mathrm{Sp}_n(\boldsymbol{Q})$, the family of the normalizers in $\mathrm{Sp}_n(\boldsymbol{R})$ of the groups of Σ consists of all maximal arithmetic groups in the commensurability class of Γ. If $\Delta(\mathfrak{A}, \mathfrak{B})$ lies in Σ, then the index of $\Delta(\mathfrak{A}, \mathfrak{B})$ in the normalizer of it in $\mathrm{Sp}_n(\boldsymbol{R})$ is 2^a, where a is the number of primes dividing $\mathfrak{J}$.*

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