BOOK REVIEW

Probability and potentials, by Paul A. Meyer. Blaisdell, New York, 1966. xiii+266 pp. $12.50.

It is difficult to explain briefly what this book is. It is much easier to explain what it is not and what it does not pretend to be. It is not a textbook, nor a book on probability theory as a whole, and not at all a book on classical potential theory. What the book wants to be and really is, is very well explained by the author's own words borrowed from his Introduction:

"The fundamental work of Doob and Hunt has shown, during the last ten years or so, that a certain form of potential theory (the study of kernels which satisfy the "complete maximum principle") and a certain branch of probability theory (the study of Markov semigroups and processes) in reality constitute a single theory. It is not a purely formal matter. Probabilistic methods have led to a much better understanding of certain fundamental ideas of potential theory (e.g. balayage, thinness, polar sets); they have above all led to a host of new results in potential theory. In turn, probability theory has received comparable mathematical advantages from this association, and a very important psychological benefit: a marked enlargement of its public, and the end of an old isolation of twenty or thirty years.

"Because of this isolation, a probabilistic background has been lacking in a number of mathematicians to whom probabilistic methods could be of great service. One can thus imagine the usefulness of a work, intended for researchers rather than students, which might put at their disposal simultaneously the elements of probability theory and some of its more advanced aspects. This need is the raison d'être of the present book."

The book consists of three parts which are "connected by a pattern of analogies rather than by explicit logical relations." But as we will see the author succeeds perfectly in making clear to his reader the interplay between probabilistic and potential theoretic notions and procedures.

Part A is entitled "Introduction to Probability Theory." Assuming that the reader is familiar with the fundamental facts of measure theory, Chapter I, "σ-Fields and Random Variables," introduces the usual probabilistic vocabulary. Chapter II concerns "Probability Laws and Mathematical Expectations" and turns after a short résumé of integration theory to deeper subjects, often neglected in measure theoretic books, like uniform integrability and the Dunford-Pettis theorem on weakly compact subsets of L^1-spaces. The fundamental facts about independence and conditioning are treated. "Complements to Measure Theory" is the content of Chapter III. This concerns, above all, the theory of analytic sets and capacities presented here in the so-called abstract form where the open sets of a topology on a set E are replaced by an arbitrary family of subsets of E containing at least the empty set. The natural character of this theory in probability is shown by a short discussion of Blackwell spaces. The usefulness for measure theory becomes clear by deriving from Choquet's famous capacitability theorem—the first theorem of the book with potential theoretic origin—a proof of the Daniell representation theorem. Regularity of finite Borel measures on a Polish space is deduced from the fact that such a space is analytic in every separable compactification. "Stochastic Processes" are introduced in Chapter IV with all necessary preliminaries, including Kolmogorov's theorem on projective limits of probability laws and a thorough discussion of separability. The fundamental notion of a stopping time is introduced. For progressively measurable processes the hitting time of an analytic set is proved to be a stopping time by means of Choquet's capacitability theorem.

Part B is exclusively devoted to "Martingale Theory." Chapter V, "Generalities and the Discrete Case," presents the now classical convergence theorems for supermartingales having the set $\boldsymbol{N}$ of natural numbers as index set via optional sampling and Doob's inequalities on the number of up- and down-crossings. Finally the first of those analogies with the classical theory of harmonic and superharmonic functions mentioned earlier appears, namely a Riesz decomposition of a supermartingale $(X_n)_{n \in \boldsymbol{N}}$ which is minorized by a submartingale. Such a supermartingale can be decomposed in the form $X_n = Y_n + Z_n$, where (Y_n) is a martingale and (Z_n) is a potential, i.e. a supermartingale with expectations $E(Z_n)$ converging to zero. (Y_n) and (Z_n) are then uniquely determined. Chapter VI extends these results to "Continuous Parameter Martingales." The discussion is limited to supermartingales $(X_t)_{t \in \boldsymbol{R}_+}$ having $\boldsymbol{R}_+$ as index set and having right continuous paths. Applications concern an analogue of the "minimum principle" of potential theory and envelopes of increasing sequences of supermartingales. Chapter VII, "Generation of Supermartingales," contains, above all, some of the author's major contributions to martingale theory. For a discrete supermartingale $(X_n)_{n \in \boldsymbol{N}}$ an old observation of Doob says that each X_n can be written in the form $X_n = Y_n + A_n$ such that (Y_n) is a martingale and (A_n) an increasing stochastic process, i.e. a process with only increasing paths $n \to A_n(\omega)$.

For right continuous supermartingales $(X_t)_{t\in R_+}$ such a Doob decomposition is only possible for a special class (DL) of supermartingales. The existence and uniqueness proof of the Doob decomposition uses a fair amount of the machinery on stochastic processes. The increasing processes give the author the opportunity to discuss the preliminaries of the theory of additive functionals. In connection with the uniqueness of the Doob decomposition, Meyer's theory about classification of stopping times is presented. In Chapter VIII the reader is rewarded for his patience by "Applications of Martingale Theory." Only nonstandard applications of the convergence theorems are given; some of them are spectacular because the use of martingales is quite unexpected. We find Doob's proof of the Hewitt-Savage theorem on permutation-symmetric laws on a product of countable many copies of a measurable space and the Choquet-Deny theorem on τ-harmonic functions on a locally compact abelian group G. The latter theorem states that a bounded continuous real-valued function h on G satisfies $h(x)=\int h(x+y)\tau(dy)$ for all $x\in G$ and for a fixed probability Radon measure τ on G if all points of the support of τ are periods for h. Also a martingale theoretic proof of the Radon-Nikodym and a part of Ionescu Tulcea's lifting theorem are given. Applications of the general theory of processes concern stochastic intervals and decompositions for square-integrable martingales.

The character of the book changes considerably in Part C, "Analytic Tools of Potential Theory." This part is the one closest to potential theory in the more classical sense of the word. Chapter IX introduces the notion of a kernel N on a measurable space $(E, \mathcal{E})$. N is by definition a mapping $N:E\times\mathcal{E}\to\overline{R}_+$ such that $A\to N(x, A)$ is always a measure on $\mathcal{E}$ and $x\to N(x, A)$ is always $\mathcal{E}$-measurable. N operates then either on the measurable functions $f\geqq 0$ on E or on the measures $\mu\geqq 0$ on $\mathcal{E}$ through the formulas

$$Nf(x) = \int N(x, dy)f(y), \quad \mu N(A) = \int \mu(dx)N(x, A).$$

Classical potential theory can be viewed as the problem of studying all measurable functions f on a measurable space $(E, \mathcal{E})$ satisfying $N_if\leqq f$ for a given family $(N_i)_{i\in I}$ of kernels on E. Only the case of a single kernel N is developed here. The measurable functions $f\geqq 0$ satisfying $Nf\leqq f$ are then called excessive; invariant functions f satisfy $Nf=f$ and are finite. Potentials are functions of the form Gh where G is the kernel $\sum_{p=0}^{\infty} N^p$ and $h\geqq 0$ is measurable. The presentation culminates in a Riesz decomposition theorem for excessive functions where the invariant functions replace the harmonic ones. Also the dual aspect is treated where excessive functions are replaced by excessive measures $\mu \geqq 0$, i.e. measures satisfying $\mu N \leqq \mu$ (and not $\mu N \leqq N$ as defined in the book). Chapter X, "Construction of Resolvents and Semigroups," is almost exclusively devoted to the famous result of Hunt which permits for a given nicely behaving kernel V on a locally compact space with countable base, with V satisfying the so-called complete maximum principle, the construction of a Feller semigroup $(P_t)_{t \in R_+}$ of kernels such that $V = \int_0^\infty P_t dt$. In this context a proof of the Hille-Yosida theorem is also given.

The final Chapter XI, "Convex Cones and Extremal Elements," is devoted, above all, to Choquet's celebrated theorem on the integral representation of convex compact sets by means of their extreme points. This fits very well into this part of the book since balayage of measures leads to a potential theoretic interpretation of the maximal measures in the Bishop-de Leeuw order relation. This order plays the central role in the proof of Choquet's theorem given in the non-metrizable case by Choquet and Meyer. Also caps, simplexes, the unicity theorem of Choquet, and the theorem of Cartier about a kernel theoretic interpretation of the Bishop-de Leeuw order are treated here. A last aspect of general potential theory appears in form of convex cones $\mathcal{S}$ of continuous functions on a compact space X. The cone $\mathcal{S}$ is mostly assumed to be inf-stable. The Choquet-Deny theorem on the measure theoretic characterization of such cones and a discussion of the Shilov boundary for $\mathcal{S}$ are included. The chapter closes with a proof of Strassen's "decomposition theorem" of linear forms bounded from above by sublinear functions. This theorem allows the construction of so-called dispersion-kernels.

The presentation in the book is always extremely clear and elegant, so elegant that parts of the book can easily be taught in a course. A small number of errors or misprints have been observed. They do not affect the overwhelming impression of the reader who finds himself rewarded with a wealth of material never covered before in book form and with the correct feeling that the applications of the results obtained here go far beyond the indications given in the book. The reader will therefore be eager to see a continuation of Professor Meyer's book in the direction of Markov processes and Hunt's potential theory (beyond the form of lecture notes).

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