PROOF. Let $G=\sum\{G_n \mid n\in J\}$ where G_n is solvable of radical class n. Then $G\in\mathfrak{B}$ and has radical class ω. Let $H=\prod\{H_k \mid k\in J,\ H_k\simeq G\}$. H has a subgroup satisfying the hypothesis of Theorem 3. Hence $H\notin\mathfrak{L}$. Consequently, $H\notin\mathfrak{B}$.

Classes of groups satisfying the conditions of Theorems 4 and 5 include the classes SN^*, SI^*, subsolvable and polycyclic.

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ALGEBRAIZATION OF ITERATED INTEGRATION ALONG PATHS[1]

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If Ω is the vector space of C^∞ 1-forms on a C^∞ manifold M, then iterated integrals along a piecewise smooth path $\alpha\colon [0,l]\to M$ can be inductively defined as below:

For $r\geqq 2$ and $w_1, w_2, \cdots, \in\Omega$,

$$\int_\alpha w_1\cdots w_r=\int_0^l\left(\int_{\alpha^t} w_1\cdots w_{r-1}\right)w_r(\alpha(t),\dot{\alpha}(t))\,dt$$

where $\alpha^t=\alpha\mid[0,t]$. (See [3].)

This note is based on the following algebraic properties of the iterated integration:

(a) $(\int_\alpha w_1\cdots w_r)(\int_\alpha w_{r+1}\cdots w_{r+s})=\sum\int_\alpha w_{\sigma(1)}\cdots w_{\sigma(r+s)}$ summing over all (r,s)-shuffles, i.e. those permutations σ of $\{1,\cdots,r+s\}$ with $\sigma^{-1}(1)<\cdots<\sigma^{-1}(r)$, $\sigma^{-1}(r+1)<\cdots<\sigma^{-1}(r+s)$.

(b) If $p=\alpha(0)$ and if f is any C^∞ function on M, then

$$\int_\alpha fw=\int_\alpha (df)w+f(p)\int_\alpha w.$$

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(c) If β is a piecewise smooth path starting from the end point of α, then

$$\int_{\alpha\beta} w_1 \cdots w_r = \int_\beta w_1 \cdots w_r + \int_\alpha w_1 \int_\beta w_2 \cdots w_r + \cdots + \int_\alpha w_1 \cdots w_r.$$

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1. Let K be a commutative unitary ring and Ω a K-module. Elements of Ω will be denoted by $w, w_1, w_2, \cdots$. Let $T(\Omega) = \oplus_{r\geq 0} T^r(\Omega)$ be the tensor K-algebra based on Ω. For $u, v \in T(\Omega)$, we shall write $uv = u \otimes v$.

Define the shuffle multiplication $\circ$ of $T(\Omega)$ by $(w_1 \cdots w_r) \circ (w_{r+1} \cdots w_{r+s}) = \sum w_{\sigma(1)} \cdots w_{\sigma(r+s)}$ summing over all (r, s)-shuffles σ. Under the shuffle multiplication, $T(\Omega)$ becomes a commutative unitary K-algebra denoted by Sh (Ω). (See [6].) Moreover Sh (Ω) has a comultiplication ζ given by

$$\zeta(w_1 \cdots w_r) = \sum_{0 \leq i \leq r} (w_1 \cdots w_i) \otimes (w_{i+1} \cdots w_r).$$

Here we set $w_1 \cdots w_r = 1$ when $r = 0$. Let $\epsilon \in \mathrm{Hom}_K(T(\Omega), K)$ be such that $\epsilon 1 = 1$ and $\epsilon T^r(\Omega) = \{0\}$ for $r \geqq 1$. With the comultiplication ζ and the counit ϵ, Sh (Ω) is a Hopf K-algebra which may be taken as a dualization of the tensor (Hopf) algebra with the diagonal map as comultiplication.

2. For any commutative unitary K-algebra A, it will be required that the canonical map $K \to A$ is injective. For any A-module Ω, it will be required that $1w = w$. We say that $d \in \mathrm{Hom}_K(A, \Omega)$ is a differentiation (of A) if $d(fg) = gdf + fdg$, $\forall f, g \in A$. If A' is also a commutative unitary K-algebra, denote by Alg (A, A') the set of morphisms $A \to A'$ of unitary K-algebras.

Denote by $\mathfrak{D}$ the category of "pointed" differentiations of K-algebras: The objects of $\mathfrak{D}$ are pairs (d, p), where $d: A \to \Omega$ is a differentiation and $p \in$ Alg (A, K). If (d', p') with $d': A' \to \Omega'$ is also an object of $\mathfrak{D}$, the set of morphisms $(d, p) \to (d', p')$ will be denoted by Diff $(d, p; d', p')$ which consists of the pairs $(\bar{\phi}, \hat{\phi})$, $\bar{\phi} \in$ Alg (A, A'), $\hat{\phi} \in \mathrm{Hom}_K(\Omega, \Omega')$ such that $\hat{\phi} d = d' \bar{\phi}$, $\hat{\phi}(fw) = (\bar{\phi} f)(\hat{\phi} w)$, $\forall f \in A$, $w \in \Omega$, and $p = p' \bar{\phi}$.

3. For any K-module Ω, one may regard Sh $(\Omega)\otimes\Omega$ as an Sh (Ω)-module. Define $\delta=\delta(\Omega)$: Sh $(\Omega)\to$Sh $(\Omega)\otimes\Omega$ such that $\delta 1=0$ and $\delta(w_1\cdots w_r)=(w_1\cdots w_{r-1})\otimes w_r$, $r\geqq 1$. Then δ is a surjective differentiation, and Sh $(\Omega)=\ker\epsilon\oplus\ker\delta$. Write $\epsilon=\epsilon(\Omega)$. The pair (δ,ϵ) can be characterized by the next theorem.

THEOREM 1. *Let (d', p') with $d'\colon A'\to\Omega'$ be an object of $\mathfrak{D}$ such that d' is surjective and $A'=\ker d'\oplus\ker p'$. Then, given any $\theta\in\mathrm{Hom}_K(\Omega,\Omega')$, there exists a unique $(\tilde\theta^\#, \hat\theta^\#)\in$* Diff $(\delta, \epsilon; d', p')$ *such that $\theta=\hat\theta^\#\iota$, where $\iota\colon\Omega\to$*Sh $(\Omega)\otimes\Omega$ *is given by $\iota(w)=1\otimes w$.*

4. An ideal J of A is said to be a d-ideal if $dJ=AdJ+J\Omega$. If J is a d-ideal, then d induces a differentiation $d_J\colon A/J\to\Omega/dJ$.

PROPOSITION. *Let $p\in$*Alg (A, K). *If $I=I(d,p)$ is the K-submodule of* Sh (Ω) *generated by $u(fw)v-(u\circ df)wv-(pf)uwv$, $\forall u, v\in$*Sh (Ω), *$w\in\Omega$, $f\in A$, then I is the smallest δ-ideal of* Sh(Ω) *that contains all $fw-(df)w-(pf)w$.*

It follows that δ induces a surjective differentiation $\Delta=\Delta(d,p)$: Sh $(\Omega)/I\to$Sh $(\Omega)\otimes\Omega/\delta$I. On the other hand, ϵ induces $\mathrm{E}=\mathrm{E}(d,p)\in$Alg (Sh $(\Omega)/I$, K) such that Sh $(\Omega)/I=\ker\Delta\oplus\ker\mathrm{E}$. The pair (Δ,E) can be characterized by the next theorem.

THEOREM 2. *Let*

$$(\tilde\chi,\hat\chi)=(\tilde\chi(d,p),\hat\chi(d,p))\in\mathrm{Diff}(d,p;\Delta,\mathrm{E})$$

be given by $\tilde\chi f=pf+df+I$, $\forall f\in A$, and $\hat\chi w=1\otimes w+\delta I$. If (d', p') is as given in Theorem 1, *then, for any $(\tilde\theta,\hat\theta)\in$*Diff $(d, p; d', p')$, *there exists one unique $(\tilde\Theta,\hat\Theta)\in$*Diff $(\Delta,\mathrm{E}; d', p')$ *such that $(\tilde\theta,\hat\theta)=(\tilde\Theta\tilde\chi,\hat\Theta\hat\chi)$.*

5. DEFINITION. A d-path from p is an element $\alpha\in$Alg (Sh (Ω), K) such that $\alpha(I)=0$. The end point of α is $q\in$Alg (A, K) given by $qf=pf+\alpha(df)$, $\forall f\in A$.

Recall that ζ is the comultiplication of Sh (Ω). For α, β $\in$Alg (Sh (Ω), K), define $\alpha\beta=(\alpha\otimes\beta)\zeta$. Then $\alpha\epsilon=\epsilon\alpha=\alpha$. It can be shown that Alg (Sh (Ω), K) is a group under the above multiplication.

THEOREM 3. *If α and β are d-paths from p to q and from q to q' respectively, then $\alpha\beta$ is a d-path from p to q'; and α^{-1} is a d-path from q to p.*

6. We say that A is d-connected if, for any $p, q\in$Alg (A, K), there exists a d-path from p to q.

PROPOSITION. *If A is d-connected and if $p, q\in$*Alg (A, K), *then $(\Delta(d,p),\mathrm{E}(d,p))\cong(\Delta(d,q),\mathrm{E}(d,q))$ in the category $\mathfrak{D}$.*

PROPOSITION. *If* Alg (A, K) *and* Alg (A', K) *are both nonempty, then* $A \oplus A'$ *is not* $(d \oplus d')$-connected.

There is a partial converse to the above assertion which states that if Alg (A, K) is the disjoint union of two nonempty sets such that there exists no d-path with its initial point in one of the sets and its end point in the other, then, under reasonable conditions, A is nontrivially imbedded in a direct sum.

PROPOSITION. *If A is d-connected with nonempty* Alg (A, K) *and if d is surjective, then A is a d-tree, i.e. A has no closed d-path other than* ϵ.

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