ON DIRECT PRODUCTS OF GENERALIZED SOLVABLE GROUPS

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Let G_α ($\alpha \in \Gamma$) be a set of groups. The direct product $\prod\{G_\alpha \mid \alpha \in \Gamma\}$ is the set of all functions f on Γ such that $f(\alpha) \in G_\alpha$ for all $\alpha \in \Gamma$, with multiplication of functions defined componentwise. The direct sum $\sum\{G_\alpha \mid \alpha \in \Gamma\}$ is the subgroup of $\prod\{G_\alpha \mid \alpha \in \Gamma\}$ consisting of all functions f with finite support.

A collection $\mathfrak{B}$ of groups is called a class of groups if $E \in \mathfrak{B}$, and isomorphic images of $\mathfrak{B}$ groups are $\mathfrak{B}$ groups. We use the following notation of P. Hall [1]. If $\mathfrak{B}$ is a class of groups, $S(\mathfrak{B})$, $Q(\mathfrak{B})$, $DS(\mathfrak{B})$, $DP(\mathfrak{B})$ denote respectively the classes of groups which are subgroups, quotient groups, direct sums and direct products of $\mathfrak{B}$ groups.

The following theorem was proved by Merzulakov in [2].

THEOREM 1. *If $\mathfrak{B}$ is a class of groups satisfying*

(a) $S(\mathfrak{B}) = \mathfrak{B}$,

(b) $Q(\mathfrak{B}) = \mathfrak{B}$,

(c) *G is a finite $\mathfrak{B}$ group if and only if G is nilpotent, then* $DP(\mathfrak{B}) \neq \mathfrak{B}$.

In this paper, a similar theorem is obtained for generalized solvable groups. Before stating these results, we need several definitions.

DEFINITION 1. Let G be a group, $x \in G$, $g \in G$. Define $[g, 0x] = g$, and inductively $[g, nx] = [[g, (n-1)x], x]$ for each positive integer n. x is called a left G Engel element if for each $g \in G$ there exists an integer $n = n(g)$ such that $[g, nx] = e$.

The Hirsch-Plotkin radical of a group G is the maximum normal locally nilpotent subgroup of G. We denote the Hirsch-Plotkin radical of G by $\phi_1(G)$.

DEFINITION 2. Let G be a group and $\phi_0(G) = E$. If α is not a limit ordinal, define $\phi_\alpha(G)$ by $\phi_\alpha(G)/\phi_{\alpha-1}(G) = \phi_1(G/\phi_{\alpha-1}(G))$. If α is a limit ordinal, define $\phi_\alpha(G)$ by $\phi_\alpha(G) = \bigcup\{\phi_\beta \mid \beta < \alpha\}$. If for some ordinal σ, $\phi_\sigma(G) = G$, G is called an LN-radical group.

In the following, $\mathfrak{L}$ will denote the class of LN-radical groups. If $G \in \mathfrak{L}$, and σ is the least ordinal for which $\phi_\sigma(G) = G$, σ is called the radical class of G. It is well known that $S(\mathfrak{L}) = \mathfrak{L}$, $Q(\mathfrak{L}) = \mathfrak{L}$, and that every solvable group is in $\mathfrak{L}$ [3]. It is easily shown that if n is a positive integer, there exist finite solvable groups of radical class n [4, p. 220].

We need the following theorem of Plotkin [3].

THEOREM 2. *If $G\in\mathfrak{L}$, then the set of left Engel elements of G is a subgroup, and this subgroup coincides with the Hirsch-Plotkin radical of G.*

In the remainder of this paper, J will denote the set of nonnegative integers.

THEOREM 3. *Let $n\in J$ and $G_n\in\mathfrak{L}$ have radical class n. Then $G=\prod\{G_n|n\in J\}\notin\mathfrak{L}$.*

PROOF. Let $R_k=\prod\{\phi_k(G_n)|n\in J\}$ and $R=\bigcup\{R_k|k\in J\}$. Then $R\lhd G$ and $R\neq G$. We show that $\phi_1(G/R)=E$.

Suppose to the contrary that $\phi_1(G/R)\neq E$ and let $yR\in\phi_1(G/R)$ with $y\notin R$. Then yR is a left G/R Engel element. Thus for each $x\in G\backslash R$, there exists a positive integer $n=n(x)$ such that $[x, ny]\in R$. Hence for each $x\in G\backslash R$, there exist nonnegative integers $n=n(x)$ and $k=k(x)$ such that $[x, ny]\in R_k$.

We now construct an $x\in G$ for which the above assertions do not hold. Since $y\notin R$, there exists $i_1\in J$ such that $y(i_1)\notin\phi_1(G_{i_1})$. By Theorem 2, $y(i_1)$ is not a left G_{i_1} Engel element. Hence there exists $x_{i_1}\in G_{i_1}$ such that $[x_{i_1}, sy(i_1)]\notin\phi_0(G_{i_1})=E$ for all $s\in J$.

Suppose nonnegative integers $i_1<i_2<\cdots<i_r$ and elements $x_{i_j}\in G_{i_j}$ $(1\leqq j\leqq r)$ have been found so that for $1\leqq j\leqq r$, $[x_{i_j}, sy(i_j)]\notin\phi_{j-1}(G_{i_j})$ for all $s\in J$. Since $y\notin R$, there exists an integer $i_{r+1}>i_r$ such that $y(i_{r+1})\notin\phi_{r+1}(G_{i_{r+1}})$. Thus, by Theorem 2 $y(i_{r+1})\phi_r(G_{i_{r+1}})$ is not a left $G_{i_{r+1}}/\phi_r(G_{i_{r+1}})$ Engel element. Hence there exists $x_{i_{r+1}}\in G_{i_{r+1}}$ such that $[x_{i_{r+1}}, sy(i_{r+1})]\notin\phi_r(G_{i_{r+1}})$ for all $s\in J$.

Let $I=\{i_1, i_2, \cdots, i_r, \cdots\}$. Define $x\in G$ as follows: $x(\eta)=x_\eta$ if $\eta\in I$ and $x(\eta)=e$ otherwise. Let $k\in J$. Then $[x, sy]\notin R_k$ for all $s\in J$. This is contrary to the first paragraph of this proof.

THEOREM 4. *Let $\mathfrak{B}$ be a class of groups such that*

(a) $\mathfrak{B}\subset\mathfrak{L}$,

(b) *every finite solvable group is contained in* $\mathfrak{B}$.

Then $DP(\mathfrak{B})\neq\mathfrak{B}$.

PROOF. The proof follows from Theorem 3 and the existence of finite solvable groups of radical class n for each $n\in J$.

The direct product $\prod\{G_\alpha|\alpha\in\Gamma\}$ is called a direct power of H if each G_α is isomorphic to H. If $\mathfrak{B}$ is a class of groups, $dp(\mathfrak{B})$ will denote the class of groups which are direct powers of $\mathfrak{B}$ groups.

In the next theorem, $\mathfrak{S}$ will denote the class of solvable groups.

THEOREM 5. *If $\mathfrak{B}$ is a class of groups such that*

(a) $\mathfrak{B}\subset\mathfrak{L}$,

(b) $DS(\mathfrak{S})\subset\mathfrak{B}$,

Then $dp(\mathfrak{B})\neq\mathfrak{B}$.

PROOF. Let $G=\sum\{G_n \mid n\in J\}$ where G_n is solvable of radical class n. Then $G\in\mathfrak{B}$ and has radical class ω. Let $H=\prod\{H_k \mid k\in J, H_k\simeq G\}$. H has a subgroup satisfying the hypothesis of Theorem 3. Hence $H\notin\mathfrak{L}$. Consequently, $H\notin\mathfrak{B}$.

Classes of groups satisfying the conditions of Theorems 4 and 5 include the classes SN^*, SI^*, subsolvable and polycyclic.

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ALGEBRAIZATION OF ITERATED INTEGRATION ALONG PATHS[1]

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If Ω is the vector space of C^∞ 1-forms on a C^∞ manifold M, then iterated integrals along a piecewise smooth path α: $[0, l]\to M$ can be inductively defined as below:

For $r\geqq 2$ and $w_1, w_2, \cdots, \in\Omega$,

$$\int_\alpha w_1\cdots w_r=\int_0^l\left(\int_{\alpha^t} w_1\cdots w_{r-1}\right)w_r(\alpha(t), \dot\alpha(t))dt$$

where $\alpha^t=\alpha\mid[0,t]$. (See [3].)

This note is based on the following algebraic properties of the iterated integration:

(a) $(\int_\alpha w_1\cdots w_r)(\int_\alpha w_{r+1}\cdots w_{r+s})=\sum\int_\alpha w_{\sigma(1)}\cdots w_{\sigma(r+s)}$ summing over all (r,s)-shuffles, i.e. those permutations σ of $\{1,\cdots,r+s\}$ with $\sigma^{-1}(1)<\cdots<\sigma^{-1}(r)$, $\sigma^{-1}(r+1)<\cdots<\sigma^{-1}(r+s)$.

(b) If $p=\alpha(0)$ and if f is any C^∞ function on M, then

$$\int_\alpha fw=\int_\alpha (df)w+f(p)\int_\alpha w.$$

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