A LIE PRODUCT FOR THE COHOMOLOGY OF SUBALGEBRAS WITH COEFFICIENTS IN THE QUOTIENT

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1. Outline. We consider an *algebra* (i.e. an associative algebra or a Lie algebra) A and a subalgebra B. Then B, A and also A/B are (two-sided) B-modules in the obvious fashion. The exact sequence of coefficient modules

$$0 \to B \xrightarrow{i} A \xrightarrow{\pi} A/B \to 0$$

induces on the (graded) Hochschild [resp. Eilenberg-Mac Lane] cohomology modules the exact triangle of homomorphisms

$$\text{(1)} \qquad \begin{array}{ccc} H^*(B, B) & \xrightarrow{\quad i^* \quad} & H^*(B, A) \\ & \nwarrow \delta^* \qquad \pi^* \swarrow & \\ & H^*(B, A/B) & \end{array}$$

The product operation in B, and similarly in A, induces a graded Lie algebra (GLA) structure (here called the *cup structure*) on $H^*(B, B)$ and $H^*(B, A)$ (cf., e.g., Gerstenhaber [2], Nijenhuis and Richardson [6]), and i^* is known to be a homomorphism of these structures. The cup structure on $H^*(B, B)$ is abelian; cf. [2]. It is also known that $H^*(B, B)$ has another GLA structure (here called the *comp structure*) with respect to the reduced grading (elements of $H^n(B, B)$ have reduced degree $n-1$; cf. [2], [7]). The following theorem supplements this information.

THEOREM. *Let A be an algebra, B a subalgebra and let A/B have its natural structure as a B-module. Then $H^*(B, A/B)$ has a GLA structure (cup structure). The maps i^* and π^* in the exact triangle* (1) *are homomorphisms of cup structures. The image of i^* belongs to the center of $H^*(B, A)$. The map δ^* is a homomorphism between the cup structure of $H^*(B, A/B)$ and the comp structure of $H^*(B, B)$.*

The theorem has immediate relevance for the theory of deformations. $H^1(B, A)$ is the set of infinitesimal nontrivial deformations of

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the homomorphism i; the cup product provides the obstructions to finite deformations (cf. Nijenhuis and Richardson [6]). $H^1(B, A/B)$ is the set of nontrivial infinitesimal deformations of B as a subalgebra of A (cf. Richardson [8]). In a forthcoming paper we shall show that the cup product provides the obstructions to finite deformations. $H^2(B, B)$ is the set of nontrivial deformations of the structure of B (subject only to the condition that the structure remain of the same type—associative or Lie), and the comp product gives the obstructions to finite deformations (cf. Gerstenhaber [3] and Nijenhuis and Richardson [7].) The homomorphisms i^*, π^* and δ^* provide the natural relationships between the infinitesimal deformations of the various kinds and the obstructions.

The origin of the formula (11) which defines the cup product in $H^*(B, A/B)$ can be found in differential geometry, where it exists as an operation yielding a vector form (differential form with values which are tangent vectors) as the product of two vector forms through a process of differentiation without the intervention of any additional structure (e.g. a connection; cf. Nijenhuis [4] and Frölicher and Nijenhuis [1]). It has been extensively applied to deformations of complex structures. The present result may also have implications for the cohomology of foliations, as a foliation is a subalgebra of the Lie algebra of vector fields.

2. **Basic formulas.** Let B denote a vector space over a field k. If f and g are cochains, i.e., elements of $C^*(B, B) = \mathrm{Hom}_k(\otimes B, B)$, of degrees n resp. m, the composition product $f \bar{\circ} g$, of degree $n+m-1$, is defined by

$$(2)\quad \begin{aligned}&(f \bar{\circ} g)(x_1, \cdots, x_{n+m-1})\\ &= \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f(x_1, \cdots, x_{i-1}, g(x_i, \cdots, x_{i+m-1}), x_{i+m}, \cdots, x_{n+m-1}).\end{aligned}$$

Although generally not associative (cf. [2]),

$$(3)\quad (f \bar{\circ} g) \bar{\circ} h - f \bar{\circ} (g \bar{\circ} h) = (-1)^{(m-1)(p-1)}\{(f \bar{\circ} h) \bar{\circ} g - f \bar{\circ} (h \bar{\circ} g)\}$$

for $h \in C^p(B, B)$, this product has a commutator

$$(4)\quad [f, g]^\circ = g \bar{\circ} f - (-1)^{(m-1)(n-1)} f \bar{\circ} g$$

which defines a GLA structure (comp structure) on $C^*(B, B)$, with respect to the reduced grading. For $\mu \in C^2(B, B)$ the condition $\mu \bar{\circ} \mu = 0$ (or $[\mu, \mu]^\circ = 0$ if char $k \neq 2$) is equivalent to μ defining an associative algebra structure on B. The Hochschild coboundary operator on $C^*(B, B)$ is given by $\delta f = -[\mu, f]^\circ$.

Let A be an associative algebra with product map μ, and let B be a subalgebra. Then every $f \in C^*(B, A)$ is the restriction to B of some (not unique) $\bar{f} \in C^*(A, A)$. The restriction of $-[\mu, \bar{f}]$ to B depends on f but not on the choice of $\bar{f}$, and is $\delta f \in C^*(B, A)$. For every $f \in C^*(B, A/B)$ there is a (nonunique) $\bar{f} \in C^*(B, A)$ such that $\pi \circ \bar{f} = f$. Then $\pi \circ \delta\bar{f}$ depends on f but not the choice of $\bar{f}$, and is just $\delta f \in C^*(B, A/B)$.

If $f, g \in C^*(B, A)$ have degrees n resp. m, then $f \cup g$, of degree $n+m$, is defined by (cf. [2])

$$(5) \quad (f \cup g)(x_1, \cdots, x_{n+m}) = \mu(f(x_1, \cdots, x_n), g(x_{n+1}, \cdots, x_{n+m})).$$

As this product is associative, commutators yield a GLA structure, defined thus:

$$(6) \qquad [f, g]^\cup = f \cup g - (-1)^{mn} g \cup f.$$

The operator δ acts as a derivation of degree 1 with respect to the cup structure on $C^*(B, A)$, hence induces the cup structure on $H^*(B, A)$. Also, $\bar{\circ}\, h$ acts as a derivation of degree $p-1$. If f has values in B, then $[f, g]^\cup$ is expressible in terms of $\bar{\circ}$ (cf. [2])

$$(7) \qquad \begin{aligned} [f, g]^\cup &= (-1)^{m-1}\{(\mu \,\bar{\circ}\, g) \,\bar{\circ}\, f - \mu \,\bar{\circ}\, (g \circ f)\} \\ &= \delta g \,\bar{\circ}\, f + (-1)^n \delta(g \,\bar{\circ}\, f) - (-1)^n g \,\bar{\circ}\, \delta f. \end{aligned}$$

This provides a formula for $\delta(g \,\bar{\circ}\, f)$, and also shows that the cup structure on $H^*(B, B)$ is abelian. In fact, it shows the following:

LEMMA 2.1. *The image $i^*(H^*(B, B))$ belongs to the center of $H^*(B, A)$ with respect to the cup structure.*

A second complex, $C^*(B, B)$ Hom_k $(\Lambda B, B)$ has a composition product, usually called the hook product, defined by

$$(8) \qquad \begin{aligned} &(f \barwedge g)(x_1, \cdots, x_{n+m-1}) \\ &\quad = \sum_\sigma \mathrm{sg}\,\sigma\, f(g(x_{\sigma(1)}, \cdots, x_{\sigma(m)}), x_{\sigma(m+1)}, \cdots, x_{\sigma(n+m-1)}) \end{aligned}$$

where the sum extends over those permutations σ of $\{1, \cdots, n+m-1\}$ for which $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(n+m-1)$. Its properties are formally completely analogous to those of $f \,\bar{\circ}\, g$, e.g. (3) holds; we define $[f, g]^\circ$ as in (4); $\mu \in C^*(B, B)$ satisfies $\mu \barwedge \mu = 0$ (equivalent to $[\mu, \mu]^\circ = 0$ if char $k \neq 2$) if and only if μ defines a Lie algebra structure on B, and the Chevalley-Eilenberg coboundary is given by $\delta f = -[\mu, f]^\circ$. The product $[f, g]^\cup$ is defined by

$$(9)\quad \begin{aligned}&[f, g]^{\cup}(x_1, \cdots, x_{n+m})\\ &\qquad = \sum_{\sigma} \operatorname{sg} \sigma\, \mu(f(x_{\sigma(1)}, \cdots, x_{\sigma(n)}), g(x_{\sigma(n+1)}, \cdots, x_{\sigma(n+m)}))\end{aligned}$$

where the sum extends over those permutations σ of $\{1, \cdots, n+m\}$ for which $\sigma(1) < \cdots < \sigma(n)$ and $\sigma(n+1) < \cdots < \sigma(n+m)$. The analogue of (7) holds, too. The references in the Lie case are [**1**], [**6**], [**7**]. Lemma 2.1 holds, too.

The case when char $k=2$, or when k is a ground ring (unitary and commutative) in which division by 2 is not possible, needs separate treatment. Many details are as in [**5**]; we only comment on a few essentials. We set $Q^{\circ}(f) = f \bar{\circ} f$ (resp. $f \bar{\wedge} f$) for n even, and $Q^{\cup}(f) = f \cup f$ in the associative case, for n odd. In the Lie case we set $Q^{\cup}(f)$ equal to the sum on the right in (9), with $f=g$, $m=n$ odd, and with the extra restriction $\sigma(1) < \sigma(n+1)$. The GLA structures thus obtained are then strong in the sense of [**5**]. The operator $Q^{\cup}$ does not generally vanish on $H^*(B, B)$, however, so the cup structure on $H^*(B, B)$ is not abelian in the strong sense. This is not surprising in view of the fact that $Q^{\cup}(f) = \operatorname{Sq}(f)$ when char $k=2$ (cf. [**3**]).

3. **Proof of the theorem.** Lemma 2.1 proves the statement on the image of i^*. The statements of Lemmas 3.1–4 show the existence of a GLA ("cup") structure on $H^*(B, A/B)$. The homomorphism properties of i^* and π^* are obvious from (11); the homomorphism property of δ^* is obvious from Lemma 3.4. All statements depend on the formal properties of §2 and are made only for the associative case. In all cases A is an algebra, B a subalgebra; $n = \deg f$ and $m = \deg g$.

LEMMA 3.1. *Let A be an algebra. Then*

$$(10)\qquad [f, g] = [f, g]^{\cup} + (-1)^{n} g \bar{\circ}\, \delta f + (-1)^{mn+m+1} f \bar{\circ}\, \delta g$$

defines a GLA structure on $C^*(A, A)$. [*When* char $k=2$, *set* $Q(f) = Q^{\cup}(f) - f \bar{\circ}\, \delta f$ *for n odd and get a strong GLA structure.*]

PROOF. By tedious computation: as this lemma is not used in the following ones, the identities derived there (with $B=A$) may be used, in addition to those in §2. See also [**9**] for some further details on the operation (10).

LEMMA 3.2 *Let $f, g \in C^*(B, A/B)$; let $\delta f = 0$, $\delta g = 0$, and let $\bar{f}, \bar{g} \in C^*(B, A)$ be such that $f = \pi \circ \bar{f}$; $g = \pi \circ \bar{g}$. Then $\delta \bar{f}, \delta \bar{g}$ have values in B, and*

(11) $$[\bar{f}, \bar{g}] = [\bar{f}, \bar{g}]^{\cup} + (-1)^{n}\bar{g} \,\bar{\circ}\, \delta\bar{f} + (-1)^{mn+m+1}\bar{f} \,\bar{\circ}\, \delta\bar{g}$$

belongs to $C^*(B, A)$ *and its projection (by left-composition with* π*) on* $C^*(B, A/B)$ *depends on* $\bar{f}$, $\bar{g}$ *(given* f, g*) by no more than a coboundary.*

PROOF. Any two choices of f differ by an element ϕ of $C^n(B, B)$. Hence, by (7)

$$\begin{aligned}[\bar{f} + \phi, \bar{g}] - [\bar{f}, \bar{g}] &= [\phi, \bar{g}]^{\cup} + (-1)^{n}\bar{g} \,\bar{\circ}\, \delta\phi + (-1)^{mn+m+1}\phi \,\bar{\circ}\, \delta\bar{g} \\ &= \delta\bar{g} \,\bar{\circ}\, \phi + (-1)^{n}\delta(\bar{g} \,\bar{\circ}\, \phi) + (-1)^{n-1}\bar{g} \,\bar{\circ}\, \delta\phi \\ &\quad + (-1)^{n}\bar{g} \,\bar{\circ}\, \delta\phi + (-1)^{mn+m+1}\phi \,\bar{\circ}\, \delta\bar{g}.\end{aligned}$$

Two terms cancel; left composition with π reduces the result to $(-1)^n\delta(g \,\bar{\circ}\, \phi)$.

LEMMA 3.3. *Let* f, g, $\bar{f}$, $\bar{g}$ *be as in Lemma* 3.2, *and let* $\bar{h} \in C^{n-1}(B, A)$ *be such that* $\delta\bar{h} = \bar{f}$. *Then* $\pi \circ [\bar{f}, \bar{g}]$ *is a coboundary in* $C^*(B, A/B)$.

PROOF. By computation

$$\begin{aligned}[\bar{f}, \bar{g}] &= [\delta\bar{h}, \bar{g}]^{\cup} + (-1)^{n}g \,\bar{\circ}\, \delta\delta\bar{h} - (-1)^{mn+m}\delta\bar{h} \,\bar{\circ}\, \delta\bar{g} \\ &= \delta[\bar{h}, \bar{g}]^{\cup} - (-1)^{n-1}[\bar{h}, \delta\bar{g}]^{\cup} - (-1)^{mn+m}\delta\bar{h} \,\bar{\circ}\, \delta\bar{g} \\ &= \delta[\bar{h}, \bar{g}]^{\cup} + (-1)^{mn+m}\{\delta\bar{h} \,\bar{\circ}\, \delta\bar{g} + (-1)^{m-1}\delta(\bar{h} \,\bar{\circ}\, \delta\bar{g}) \\ &\quad + (-1)^{m+1}\bar{h} \,\bar{\circ}\, \delta\delta\bar{g}\} - (-1)^{mn+m}\delta\bar{h} \,\bar{\circ}\, \delta\bar{g}.\end{aligned}$$

Two terms cancel, one is zero, and the rest are coboundaries. Left-composition with π yields coboundaries in $C^*(B, A/B)$.

LEMMA 3.4. *Let* f, g, $\bar{f}$, $\bar{g}$ *be as in Lemma* 3.2; *then*

(12) $$\delta[\bar{f}, \bar{g}] = [\delta\bar{f}, \delta\bar{g}]^{\circ} \in C^*(B, B).$$

PROOF. By computation:

$$\begin{aligned}\delta[\bar{f}, \bar{g}] &= \delta[\bar{f}, \bar{g}]^{\cup} + (-1)^{n}\delta(\bar{g} \,\bar{\circ}\, \delta\bar{f}) + (-1)^{mn+m+1}\delta(\bar{f} \,\bar{\circ}\, \delta\bar{g}) \\ &= \delta[\bar{f}, \bar{g}]^{\cup} + (-1)^{n}\{(-1)^{n+1}[\delta\bar{f}, \bar{g}]^{\cup} + (-1)^{n}\delta\bar{g} \,\bar{\circ}\, \delta\bar{f} + \bar{g} \,\bar{\circ}\, \delta\delta\bar{f}\} \\ &\quad + (-1)^{mn+m+1}\{(-1)^{m+1}[\delta\bar{g}, \bar{f}]^{\cup} + (-1)^{m}\delta\bar{f} \,\bar{\circ}\, \delta\bar{g} + \bar{f} \,\bar{\circ}\, \delta\delta\bar{g}\} \\ &= \delta[\bar{f}, \bar{g}]^{\cup} - [\delta\bar{f}, g]^{\cup} - (-1)^{n}[\bar{f}, \delta\bar{g}]^{\cup} + \delta\bar{g} \,\bar{\circ}\, \delta\bar{f} + (-1)^{mn+1}\delta\bar{f} \,\bar{\circ}\, \delta\bar{g} \\ &= [\delta\bar{f}, \delta\bar{g}]^{\circ}.\end{aligned}$$

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BOUNDED APPROXIMATION BY POLYNOMIALS WITH RESTRICTED ZEROS[1]

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1. Introduction. Let C be a rectifiable Jordan curve, D its interior. A sequence of polynomials $P_n(z)$ is said to converge boundedly to a function $f(z)$ in D, or equivalently, $f(z)$ is said to be boundedly approximated by the polynomials $P_n(z)$ in D, if $\sup\{|P_n(z)| : z\in D\}$ is bounded as a function of n, and $\{P_n(z)\}$ converges to $f(z)$ throughout D. It is known [1], [6] that $f(z)$ can be boundedly approximated by polynomials in D if and only if $f(z)$ is a bounded holomorphic function in D. In this paper we consider the more delicate bounded approximation problem in which the zeros of the polynomials are required to lie on the boundary C. Polynomials whose zeros lie on C are called C-polynomials.

A different kind of appproximation by C-polynomials was studied by G. R. MacLane [5]. He proved that if $f(z)$ is holomorphic and zero free in D, then there exists a sequence of C-polynomials which converges to $f(z)$ uniformly on every compact subset of D. This result was later extended by J. Korevaar [3] and his students [4] to more general sets D.

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