GENERATORS FOR SOME RINGS OF ANALYTIC FUNCTIONS

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Let Ω be an open set in C^n and let p be a nonnegative function defined in Ω. We shall denote by $A_p(\Omega)$ the set of all analytic functions f in Ω such that for some constants C_1 and C_2

$$|f(z)| \leqq C_1 \exp(C_2 p(z)), \qquad z \in \Omega. \tag{1}$$

It is obvious that $A_p(\Omega)$ is a ring. We wish to determine when it is generated by a given finite set of elements $f_1, \cdots, f_N$. There is an obvious necessary condition, for if $f_1, \cdots, f_N$ are generators for $A_p(\Omega)$ we can in particular find $g_1, \cdots, g_N \in A_p(\Omega)$ so that $1 = \sum f_j g_j$. Hence we have

$$1 \leqq \sum |f_j(z)| C_1 \exp(C_2 p(z))$$

for some constants C_1 and C_2, that is,

$$|f_1(z)| + \cdots + |f_N(z)| \geqq c_1 \exp(-c_2 p(z)), \qquad z \in \Omega, \tag{2}$$

for some positive constants c_1 and c_2.

This note concerns the converse statement. Carleson [1] has proved a deep result of that type, called the Corona Theorem, which states that (2) implies that $f_1, \cdots, f_N$ generate $A_p(\Omega)$ if $p = 0$ and Ω is the unit disc in C. In a recent research announcement [5] in this Bulletin, the Corona Theorem was used to prove the analogous result when $p(z) = |z|$ and $\Omega = C$. However, we shall see here that this statement is much more elementary than the Corona Theorem; indeed, we shall prove a general result of this kind for functions of several complex variables although no analogue of the Corona Theorem is known there.

THEOREM 1. *Let p be a plurisubharmonic function in the open set $\Omega \subset C^n$ such that*

(i) *all polynomials belong to $A_p(\Omega)$;*

(ii) *there exist constants $K_1, \cdots, K_4$ such that $z \in \Omega$ and $|z - \zeta| \leqq \exp(-K_1 p(z) - K_2) \Rightarrow \zeta \in \Omega$ and $p(\zeta) \leqq K_3 p(z) + K_4$.*

Then $f_1, \cdots, f_N \in A_p(\Omega)$ generate $A_p(\Omega)$ if and only if (2) *is valid.*

Before the proof we make a few remarks. First note that if $d(z)$

denotes the distance from $z \in \Omega$ to $C\Omega$ then (ii) implies that $d(z) \geq \exp(-K_1 p(z) - K_2)$, that is,

$$p(z) \geq (\log 1/d(z) - K_2)/K_1.$$

Hence $p(z) \to \infty$ if z converges to a boundary point of Ω, so Ω is pseudoconvex and therefore a domain of holomorphy (cf. [3, Theorem 4.2.8]). On the other hand, if Ω is a domain of holomorphy it follows that $p(z) = \log 1/d(z)$ is plurisubharmonic, and (ii) is valid with $K_1 = K_3 = 1$ and suitable K_2, K_4. Another example is obtained by taking $p(z) = \sum |z_j|^p$, $\Omega = \mathbf{C}^n$, where p is any positive number. When $n = 1$ this yields the results announced in [5]. However, the Corona Theorem is not contained in Theorem 1 but will be discussed at the end of the note.

We know already that (2) is a necessary condition for $f_1, \cdots, f_N$ to be generators. To prove the sufficiency we shall apply a standard homological argument (cf. e.g. Malgrange [6]) but first a few lemmas are required.

LEMMA 2. *If $f \in A_p(\Omega)$ it follows that $\partial f/\partial z_j \in A_p(\Omega)$.*

PROOF. From (1) and (ii) we obtain

$$|f(\zeta)| \leq C_1 \exp(C_2(K_3 p(z) + K_4)) \quad \text{if } |\zeta - z| \leq \exp(-K_1 p(z) - K_2).$$

Hence

$$|\partial f(z)/\partial z_j| \leq C_1 \exp(C_2(K_3 p(z) + K_4) + K_1 p(z) + K_2).$$

Since we shall use $\bar\partial$ cohomology with bounds in L^2 norms, we also note that the definition of $A_p(\Omega)$ can be expressed in terms of such norms.

LEMMA 3. *If f is analytic in Ω, then $f \in A_p(\Omega)$ if and only if for some K*

$$\int |f|^2 e^{-2Kp} d\lambda < \infty, \tag{3}$$

where $d\lambda$ denotes the Lebesgue measure.

PROOF. If (1) is valid we obtain (3) since $(1 + |z|)^{2n+1} \leq B_1 \exp B_2 p(z)$ in view of (i). On the other hand, it follows from (3) and (ii) that the mean value of $|f|$ over the ball $\{\zeta;\ |\zeta - z| \leq \exp(-K_1 p(z) - K_2)\}$ is bounded by $C \exp(K(K_3 p(z) + K_4) + 2n(K_1 p(z) + K_2))$. Since this is also a bound for $|f(z)|$, the lemma is proved.

LEMMA 4. *Let g be a form of type $(0, r+1)$ in Ω with locally square integrable coefficients and $\bar\partial g=0$, and let ϕ be a plurisubharmonic function in Ω such that*

$$\int |g|^2 e^{-\phi} d\lambda < \infty.$$

If $r \geqq 0$ it follows that there is a form f of type $(0, r)$ with $\bar\partial f=g$ and

$$\int |f|^2 e^{-\phi}(1+|z|^2)^{-2} d\lambda \leqq \int |g|^2 e^{-\phi} d\lambda. \tag{4}$$

The norms here are defined as in §4.1 of [3]. The lemma follows from Theorem 2.2.1′ in [2] by the argument used in [3] to derive Theorem 4.4.2 from Theorem 4.4.1.

For nonnegative integers s and r we shall denote by L_r^s the set of all differential forms h of type $(0, r)$ with values in $\Lambda^s \mathbf{C}^N$, such that for some K

$$\int |h|^2 e^{-2Kp} d\lambda < \infty.$$

In other words, for each multi-index $I=(i_1, \cdots, i_s)$ of length $|I|=s$ with indices between 1 and N inclusively, h has a component h_I which is a differential form of type $(0, r)$ such that h_I is skew symmetric in I and

$$\int |h_I|^2 e^{-2Kp} d\lambda < \infty.$$

The $\bar\partial$ operator defines an unbounded map from L_r^s to L_{r+1}^s; its domain consists of all $h \in L_r^s$ such that $\bar\partial h$, defined in the sense of distribution theory with $\bar\partial$ acting on each component h_I is an element of L_{r+1}^s. Furthermore, the interior product P_f by $(f_1, \cdots, f_N)$ maps L_r^{s+1} into L_r^s: If $h \in L_r^{s+1}$ then

$$(P_f h)_I = \sum_1^N h_{Ij} f_j, \qquad |I| = s.$$

We define $P_f L_r^0 = 0$. Clearly $P_f^2=0$ and P_f commutes with $\bar\partial$ since f_j are analytic, so we have a double complex.

LEMMA 5. *The equation $\bar\partial g=h$ has a solution $g \in L_r^s$ for every $h \in L_{r+1}^s$ with $\bar\partial h=0$.*

PROOF. In view of (i) this is an immediate consequence of Lemma 4.

LEMMA 6. *If $g \in L_r^s$ and $P_f g = 0$, we can find $h \in L_r^{s+1}$ such that $g = P_f h$ and in addition $\bar\partial h \in L_{r+1}^{s+1}$ if $\bar\partial g = 0$.*

PROOF. We can take for h essentially the exterior product of g by $\bar f / |f|^2$. More precisely, we set when $|I| = s+1$

$$h_I = \sum_1^{s+1} g_{I_j} (-1)^{s1-j} \bar f_{i_j} / |f|^2,$$

where I_j denotes the multi-index $I = (i_1, \cdots, i_{s+1})$ with the index i_j removed. It follows from (2) that $h \in L_r^{s+1}$, and since $P_f g = 0$ it is obvious that $P_f h = g$. If $\bar\partial g = 0$ we can compute $\bar\partial h_I$ by operating on the factor $\bar f_{i_j} / |f|^2$ alone, so it follows from (2) and Lemma 2 that $\bar\partial h \in L_{r+1}^{s+1}$.

It is now easy to prove the following theorem which in view of Lemma 3 contains Theorem 1 for $r = s = 0$. (Actually Theorems 1 and 7 are equivalent.)

THEOREM 7. *For every $g \in L_r$ with $\bar\partial g = P_f g = 0$ one can find $h \in L_r^{s+1}$ so that $\bar\partial h = 0$ and $P_f h = g$.*

PROOF. The theorem is trivially valid when $r > n$ or $s > N$. In the proof we may therefore assume that it has already been established for larger values of r and s. By Lemma 6 we can find $h' \in L_r^{s+1}$ so that

$$P_f h' = g, \qquad \bar\partial h' \in L_{r+1}^{s+1}.$$

Since $\bar\partial \bar\partial h' = 0$ and $P_f \bar\partial h' = \bar\partial P_f h' = \bar\partial g = 0$, it follows from the inductive hypothesis that one can find $h'' \in L_{r+1}^{s+2}$ such that

$$P_f h'' = \bar\partial h', \qquad \bar\partial h'' = 0.$$

By Lemma 5 we can find $h''' \in L_r^{s+2}$ so that $\bar\partial h''' = h''$. If $h = h' - P_f h'''$ we conclude that $\bar\partial h = \bar\partial h' - P_f \bar\partial h''' = \bar\partial h' - P_f h'' = 0$, and that $P_f h = P_f h' = g$. The proof is complete.

We shall end this note by showing how the proofs of Carleson [1] can be adapted to the conventional pattern used in the proof of Theorem 1. This does not remove the main difficulties but it does eliminate a tricky argument due to D. J. Newman, which was used in [1] in the case of more than 2 generators. In the proof of Theorem 1 the main points were the existence theorems for the operators $\bar\partial$ and P_f given in Lemmas 5 and 6. The proof of the Corona Theorem requires a more precise version of both.

From now on Ω will denote the unit disc in **C**. (All the arguments are valid for any bounded open set in **C** with a C^2 boundary.) If μ is a bounded measure in Ω and ϕ is an integrable function on $\partial\Omega$, we shall say that a distribution in Ω satisfying the Cauchy-Riemann equation

$$\partial u/\partial \bar{z} = \mu \text{ in } \Omega \tag{5}$$

has boundary values ϕ on $\partial\Omega$ provided that there exists a distribution U with support in $\bar{\Omega}$ such that $U=u$ in Ω and

$$\partial U/\partial \bar{z} = \mu - \phi dz/2i. \tag{6}$$

Here ϕdz is of course a measure on $\partial\Omega$, and μ is extended so that there is no mass in the complement of Ω. If $u=0$ it follows from (6) that $U=0$, for $\partial U/\partial \bar{z}$ would otherwise be a distribution with support on $\partial\Omega$ with positive transversal order. Hence u determines both μ, ϕ and U uniquely, so it is legitimate for us to say that ϕ is the boundary value of u.

If u belongs to the Hardy class H^p for some $p \geqq 1$, then ϕ coincides a.e. with the boundary values in the usual sense, and $\mu=0$. Conversely, if u is analytic and has boundary values belonging to $L^p(\partial\Omega)$ in the sense of (6), it follows that $u \in H^p$ $(p \geqq 1)$. If $f \in H^\infty$ and u is a solution of (5) with boundary values ϕ, then fu satisfies (5) with μ replaced by $f\mu$ and has boundary values $f\phi$. This is obvious when f is analytic in a neighborhood of $\bar{\Omega}$ and follows in general if we first consider $f(rz)$ with $r<1$ and then let $r \to 1$, noting that the solution $U \in \mathcal{E}'(\bar{\Omega})$ of the equation $\partial U/\partial \bar{z} = F$ is a continuous function of $F \in \mathcal{E}'(\bar{\Omega})$ when it exists.

The existence of a solution of (6) with support in $\bar{\Omega}$ means precisely that the right hand side is orthogonal to all (entire) analytic functions. Thus (5) has a solution with boundary values ϕ if and only if for entire analytic f

$$\int f d\mu = (2i)^{-1} \int \phi(z) f(z) dz.$$

In view of the Hahn-Banach Theorem it follows that there exists a solution with boundary values of absolute value $\leqq C$ if and only if for entire analytic f

$$\left| \int f d\mu \right| \leqq C \int |f(z)| \, |dz| / 2.$$

A sufficient condition for this is given by the following result of [1]. (See also [4] where an extension to several variables is given.)

LEMMA 8. *There is a constant C such that*

$$(7)\quad \int_\Omega |v(z)|^p\, |d\mu(z)| \leqq CM \int_{\partial\Omega} |v|^p\, |dz|, \qquad v \in H^p(\Omega),\ p > 0,$$

for every measure μ *in* Ω *such that*

$$(8)\qquad |\mu|\{\zeta;\ |\zeta - z| < r\} \leqq Mr, \qquad z \in \partial\Omega,\ r > 0.$$

We now modify the definition of L_r^s as follows:

$h \in L_0^s$ if $\partial h_I/\partial\bar z$ is a bounded measure in Ω and h_I has boundary values in $L^\infty(\partial\Omega)$, $|I| = s$; $h \in L_1^s$ if $h_I = \mu_I d\bar z$ where μ_I is a measure in Ω satisfying (8), $|I| = s$. Of course we take $L_r^s = 0$ when $r > 1$. From Lemma 8 and the discussion preceding it we conclude that Lemma 5 remains valid and that $\{h;\ h \in L_0^0,\ \bar\partial h = 0\} = H^\infty$.

Let $f_j \in H^\infty$, $j = 1, \cdots, N$, and assume that for some $c > 0$

$$(2)'\qquad |f_1(z)| + \cdots + |f_N(z)| \geqq c.$$

If we define P_f by means of these functions, the proof of Lemma 6 remains valid when $s = 1$ but breaks down when $s = 0$ since $\partial f_j/\partial z$ need not be a bounded function. We must therefore use another construction, based on the following

LEMMA 9. *For sufficiently small* $\epsilon > 0$ *one can find a partition of unity* ϕ_j *subordinate to the covering of* Ω *by the open sets* $\Omega_j = \{z;\ |f_j(z)| > \epsilon\}$ *such that* $\partial\phi_j/\partial\bar z$, *defined in the sense of distribution theory, is a measure which satisfies* (8) *for all* j *and some* M.

Admitting Lemma 9 for a moment we shall see that it implies the Corona Theorem. With our new definition of L_r we have already seen that Lemma 5 remains valid as well as Lemma 6 for $r \neq 0$. To prove Lemma 6 for $r = 0$ we need only replace $\bar f_j/|f|^2$ in the previous proof by ϕ_j/f_j where ϕ_j is the partition of unity in Lemma 9. In fact, $\partial(\phi_j/f_j)/\partial\bar z = f_j^{-1}\partial\phi_j/\partial\bar z$ satisfies (8) since $|f_j| \geqq \epsilon$ in supp ϕ_j. Hence the proof of Theorem 7 can be applied without change. For $r = s = 0$ we obtain the only interesting conclusion:

THEOREM 10. (*The Corona Theorem*). *If* $f_1, \cdots, f_N \in H^\infty$ *and* (2)′ *is valid, it follows that* $f_1, \cdots, f_N$ *are generators for* H^∞.

It remains to discuss the proof of Lemma 9. Since the set of bounded functions ψ with $\partial\psi/\partial\bar z$ satisfying (8) is a ring, the standard technique for constructing partitions of unity can be applied to derive Lemma 9 from

LEMMA 11. *There exists a constant* k *such that if* $0 < \epsilon < \frac{1}{2}$ *and* $f \in H^\infty$, $\sup |f| \leqq 1$, *one can find* ψ *with* $0 \leqq \psi \leqq 1$ *so that* $\partial\psi/\partial\bar z$ *satisfies* (8) *and*

$$\psi(z) = 0 \text{ when } |f(z)| < \epsilon^k, \qquad \psi(z) = 1 \text{ when } |f(z)| > \epsilon.$$

This lemma was proved in a different formulation in [1] when f is a Blaschke product. In fact, the main point in [1] is a construction of certain curves Γ surrounding the zeros of a Blaschke product and satisfying conditions which mean precisely that the characteristic function ψ of the exterior of Γ has the properties stated in Lemma 11. Since the proof given in [1] is applicable to arbitrary $f \in H^\infty$ and we have no significant simplification to contribute, we shall not carry out the proof here.

References

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