ITERATED PATH INTEGRALS AND GENERALIZED PATHS[1]

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Let $\mathfrak{M}$ be a C^∞ manifold with a countable basis. For convenience, it is assumed that $\mathfrak{M}$ is Riemannian. Let $\mathfrak{P}$ be the set of "reduced" piecewise C^1 paths having a common initial point p in $\mathfrak{M}$ such that each $\alpha \in \mathfrak{P}$ is parameterized by arc length. By a reduced path α: $[0, l] \to \mathfrak{M}$, we mean one such that there exists no $t \in (0, l)$ with $\alpha(t-s) = \alpha(t+s)$ for $|s|$ sufficiently small.

Let Ω be the vector space (over the real number field R) of C^∞ 1-forms on $\mathfrak{M}$. Elements of Ω will be denoted by $w, w_1, w_2, \cdots$. Let α^t be the restriction $\alpha | [0, t]$, $0 \leqq t \leqq l$. Let $\int_\alpha w_1$ be the usual integral, and define, for $r > 1$,

$$\int_\alpha w_1 \cdots w_r = \int_0^l \left(\int_{\alpha^t} w_1 \cdots w_{r-1} \right) w_r(\alpha(t), \dot{\alpha}(t)) dt.$$

Each iterated integral $\int w_1 \cdots w_r$ is thus a real valued function on $\mathfrak{P}$. The totality of iterated integrals together with the constant functions on $\mathfrak{P}$ generates a subalgebra F of the R-algebra of real valued functions on $\mathfrak{P}$. The R-algebra F is of interest for two reasons: (a) Elements of F have geometrical significance of the manifold $\mathfrak{M}$. (b) It follows from results in [1] that F contains sufficiently many functions on $\mathfrak{P}$ as to separate the points of $\mathfrak{P}$.

The purpose of this note is to give some indication of the structure of F. In particular, Theorem 2 implies that, if $\mathfrak{M} = R^n$, then F contains a dense subalgebra which is algebraically isomorphic with a polynomial algebra of, at most, countably many indeterminates.

We shall also introduce the notion of a generalized path in $\mathfrak{M}$ which is obtained through a process of dualization in a manner somewhat more complicated than that of a 1-dimensional current. (See [4].) The multiplication of generalized paths is nonabelian.

A detailed account will be given in a forthcoming paper.

1. Given any compact subset K of $\mathfrak{M}$, define the seminorm $\| \ \|_K$ of Ω such that

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$$\|w\|_K = \sup\{\|w_q\| : q \in K\}$$

where $\|w_q\|$ is the length of the cotangent vector w_q. Let $T^r(\Omega) = \Omega \otimes \cdots \otimes \Omega$ be the r-fold tensor product of Ω over R. If $u_r \in T^r(\Omega)$, define $\|u_r\|_K$ to be the infimum of all $\sum_i \|w_1^{(i)}\|_K \cdots \|w_r^{(i)}\|_K$ for all possible finite sums $u_r = \sum_i w_1^{(i)} \otimes \cdots \otimes w_r^{(i)}$. Any element u of the tensor algebra $T(\Omega) = \oplus_{r \geqq 0} T^r(\Omega)$ is a finite sum $u = \sum u_r$, $u_r \in T^r(\Omega)$, where $T^0(\Omega) = R$. Given a sequence $M = (M_0, M_1, \cdots)$ of positive numbers, define

$$\|u\|_{M,K} = \sum M_r \|u_r\|_K$$

where $\|u_0\|_K = |u_0|$. Then $T(\Omega)$ becomes a locally convex R-algebra whose topology is generated by all seminorms $\| \;\|_{M,K}$.

Define, in the tensor algebra $T(\Omega)$, a bilinear multiplication $\circ$ called the shuffle multiplication such that, for $r \geqq 0$, $s \geqq 0$,

$$(w_1 \otimes \cdots \otimes w_r) \circ (w_{r+1} \otimes \cdots \otimes w_{r+s}) = \sum w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(r+s)}$$

summing over those permutations σ of the set $\{1, \cdots, r+s\}$ with $\sigma^{-1}(1) < \cdots < \sigma^{-1}(r)$, $\sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s)$. We shall assume that $w_1, \cdots, w_r = 1$ for $r = 0$. Then, under the shuffle multiplication, $T(\Omega)$ becomes a unitary commutative R-algebra $\mathrm{Sh}(\Omega)$. Since the shuffle multiplication is continuous, $\mathrm{Sh}(\Omega)$ is a locally convex R-algebra.

2. Define a linear map j: $T(\Omega) \to F$ such that $j1 = 1$ and $j(w_1 \otimes \cdots \otimes w_r) = \int w_1 \cdots w_r$. Then j is surjective. Moreover, for any u, $v \in T(\Omega)$,

$$j(u \circ v) = j(u)j(v)$$

so that j is an epimorphism from the R-algebra $\mathrm{Sh}(\Omega)$ onto the R-algebra F. (This observation is essentially due to Ree [6].)

Given $\alpha \in \mathfrak{P}$, denote by e_α: $F \to R$ the evaluation map at α. Then

$$e_\alpha j(w_1 \otimes \cdots \otimes w_r) = \int_\alpha w_1 \cdots w_r.$$

It can be shown that $e_\alpha j$: $\mathrm{Sh}(\Omega) \to R$ is continuous. Then $\ker e_\alpha j$ is a closed ideal of $\mathrm{Sh}(\Omega)$. Therefore $\ker j = \bigcap_{\alpha \in \mathfrak{P}} \ker e_\alpha j$ is a closed ideal of $\mathrm{Sh}(\Omega)$. We topologize the R-algebra F through the isomorphism $\mathrm{Sh}\,(\Omega)/\ker j \cong F$.

3. If f is a C^∞ function on $\mathfrak{M}$ and if $\alpha \in \mathfrak{P}$, then

$$\int_\alpha w_1 \cdots w_r(fw)w_{r+1} \cdots w_{r+s}$$
$$= \int_\alpha ((w_1 \cdots w_r) \circ df)ww_{r+1} \cdots w_{r+s}$$
$$+ f(p) \int_\alpha w_1 \cdots w_r w w_{r+1} \cdots w_{r+s}.$$

It follows that $\ker j$ contains I_p which is the closure of the subspace of $\mathrm{Sh}(\Omega)$ spanned by all elements of the type $u(fw)v-(u \circ df)wv + f(p)uwv$, where $u, v \in T(\Omega)$, $w \in \Omega$, and f is a C^∞ function on $\mathfrak{M}$. It can be shown that I_p is an ideal of $\mathrm{Sh}(\Omega)$. Consequently j induces a continuous epimorphism j_p: $\mathrm{Sh}(\Omega)/I_p \to F$.

For any subspace Ω_0 of Ω, the inclusion $\Omega_0 \subset \Omega$ induces a continuous homomorphism $\mathrm{Sh}(\Omega_0) \to \mathrm{Sh}(\Omega)$. If $\dim \Omega_0 = 1$, then $\mathrm{Sh}(\Omega_0)$ is isomorphic with the polynomial algebra $R(x)$; if $1 < \dim \Omega_0 < \infty$, it is known that $\mathrm{Sh}(\Omega_0) \cong R(x_1, x_2, \cdots)$.

THEOREM 1. *There exist C^∞ functions $h_1, \cdots, h_m$ on M with $\frac{1}{2}(m-1) \leqq \dim \mathfrak{M}$ such that, if Ω_0 is the subspace of Ω spanned by $dh_1, \cdots, dh_m$, then* Sh (Ω_0) *has a dense image in* Sh $(\Omega)/I_p$ *under the composite homomorphism* Sh $(\Omega_0) \to$ Sh $(\Omega) \to$ Sh $(\Omega)/I_p$.

COROLLARY. *The algebra F has a dense subalgebra which is a homomorphic image of a polynomial algebra with, at most, countably many indeterminates.*

In the case of $\mathfrak{M} = R^n$, we have an additional result.

THEOREM 2. *If Ω is the space of C^∞ 1-forms on R^n and if Ω_0 is the subspace spanned by the 1-forms $dx^1, \cdots, dx^n$, where $x^1, \cdots, x^n$ are the coordinates of R^n, then the composite homomorphism*

$$\mathrm{Sh}\ (\Omega_0) \to \mathrm{Sh}\ (\Omega) \xrightarrow{J} F$$

is injective.

4. DEFINITION. A continuous homomorphism α: Sh $(\Omega) \to R$ such that $\alpha 1 = 1$ and $\alpha(I_p) = 0$ is called a generalized path in $\mathfrak{M}$ with the initial point p.

The generalized path ϵ such that $\epsilon(T^r(\Omega)) = 0$ for all $r \geqq 1$ is called the constant generalized path. Every generalized path $\alpha \neq \epsilon$ has a unique initial point p and a unique terminal point q such that, for any C^∞ function f on $\mathfrak{M}$, $\alpha(df) = f(q) - f(p)$.

If α and β are generalized paths in $\mathfrak{M}$, we define $\alpha\beta$: Sh $(\Omega) \to R$ such that

$$\alpha\beta(w_1 \otimes \cdots \otimes w_r) = \sum_{0 \leq i \leq r} \alpha(w_1 \otimes \cdots \otimes w_i)\beta(w_{i+1} \otimes \cdots \otimes w_r).$$

THEOREM 3. *If α and β are generalized paths from p to q and from q to q' respectively, then $\alpha\beta$ is a generalized path from p to q'. Moreover, there exists a generalized path α^{-1} from q to p such that $\alpha^{-1}\alpha = \alpha\alpha^{-1} = \epsilon$.*

Obviously $\alpha\epsilon = \epsilon\alpha = \alpha$.

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