ON TAUBERIAN CONDITIONS OF TYPE o

BY WERNER MEYER-KÖNIG AND HUBERT TIETZ

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The series $\sum a_n$ ($\sum$ means $\sum_{n=0}^{\infty}$) is said to be summable to the sum s by Abel's method of summability, if $\sum a_n x^n = f(x)$ is convergent for $0<x<1$ and if $f(x) \to s$ as $x \to 1^-$ (x real). A classical theorem of A. Tauber [2] states that if $\sum a_n$ is summable to the sum s by Abel's method and if

$$(1) \qquad a_n = o(1/n) \quad \text{as} \quad n \to \infty$$

then $\sum a_n = s$. In today's language we put this in the following way: (1) is a Tauberian condition for Abel's method (cf., e.g., Hardy [1, pp. 149–152]). Again according to Tauber [2] the weaker condition

$$(2) \qquad \delta_n = o(1) \quad \text{with} \quad \delta_n = (n+1)^{-1} \sum_{k=0}^{n} k a_k$$

is also a Tauberian condition for Abel's method.

We shall show that Tauber's passage from (1) to (2) is possible for a very general class of summability methods. Formula (3) which yields this passage was already used by Tauber [2, p. 276, (6)]; here we exploit it more fully.

The summability method V is said to be regular if $\sum a_n = s$ implies $V\text{-}\sum a_n = s$. V is called additive if

$$V\text{-}\sum a_n = s, \qquad V\text{-}\sum b_n = t \text{ implies } V\text{-}\sum (a_n + b_n) = s + t.$$

THEOREM. *If* (1) *is a Tauberian condition for the regular and additive method V then also* (2) *is a Tauberian condition for V.*

PROOF. We assume that (1) is a Tauberian condition for V and that we have under consideration a given series $\sum a_n$ which is summable V to the sum s and for which (2) is fulfilled. We have to show that $\sum a_n = s$. Putting $b_0 = a_0$ and $b_n = \delta_n / n$ ($n = 1, 2, \cdots$) the equation

$$(3) \quad a_0 + \cdots + a_n = (b_0 + \cdots + b_n) + \delta_n \qquad (n = 0, 1, \cdots)$$

is easily proved by induction. Together with $V\text{-}\sum a_n = s$ and $V\text{-}\lim(-\delta n) = 0$, (3) gives $V\text{-}\sum b_n = s$. Since $b_n = o(1/n)$ we conclude that $\sum b_n = s$ and further, again from (3), that $\sum a_n = s$.

If, a sequence λ being given ($\lambda_{n-1}<\lambda_n\rightarrow\infty$ as $n\rightarrow\infty$), (1) is replaced by

$$\text{(1a)} \qquad a_n = o(1/\lambda_n) \quad \text{as} \quad n \rightarrow \infty$$

and (2) by

$$\text{(2a)} \qquad (n+1)^{-1}(\lambda_0 a_0 + \cdots + \lambda_n a_n) = o(1),$$

the theorem is still true provided that

$$n/\lambda_n = O(1) \quad \text{and} \quad n(\lambda_{n+1} - \lambda_n)/\lambda_{n+1} = O(1).$$

Herewith the cases

$$\lambda_n = n \log n, \qquad \lambda_n = n \log n \log\log n, \cdots$$

are covered. The theorem fails to remain true if $n/\lambda_n\rightarrow\infty$. A paper investigating these questions and similar ones is under preparation.

References

1. G. H. Hardy, *Divergent series*, Clarendon Press, Oxford, 1949.
2. A. Tauber, *Ein Satz aus der Theorie der unendlichen Reihen*, Monatsh. Math. Phys. 8 (1897), 273–277.

Technische Hochschule Stuttgart, West Germany