COMPACTIFICATION OF STRONGLY COUNTABLE DIMENSIONAL SPACES[1]

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In this paper all spaces, including compactifications, are separable metrizable. Recall the following definitions. A space X is strongly countable dimensional if X is a countable union of closed finite-dimensional subsets. X is a G_δ space if X is a G_δ-set in each space in which it is topologically embedded. A space Y is a pseudo-polytope if $Y=\Sigma_1\cup\Sigma_2\cup\cdots$, where each Σ_i is a simplex, $\Sigma_i\cap\Sigma_j$ is either empty or a face of both Σ_i and Σ_j, and diam $\Sigma_i\to 0$ as $i\to\infty$. The term map always denotes a continuous function. Other notation is as in [3] and [8].

In [5] Lelek proved that every G_δ-space X has a compactification dX such that $dX\backslash X$ is a pseudo-polytope. He then raised the question of whether every strongly countable dimensional G_δ space X has a strongly countable dimensional compactification. This paper answers that question in the affirmative. We first state some preliminary propositions.

PROPOSITION 1. *Let $M\subset X$ with* dim $M\leqq n$, *and let* $\{U_i \mid i=1,2,\cdots\}$ *be a sequence of sets open in X and covering M. Then there is a sequence* $\{V_i\mid i=1,2,\cdots\}$ *of sets open in X and covering M such that* ord$\{V_i\mid i=1,2,\cdots\}\leqq n+1$ *and such that* $V_{k(n+1)+j}\subset U_{k+1}$ *for* $k=0,1,2,\cdots$ *and* $j=1,2,\cdots,n+1$.

PROOF. The proof involves only a slight extension of the argument on page 54 of [2].

PROPOSITION 2. *Let G be an open subset of a totally bounded space Y, and let $M_1, M_2,\cdots, M_r$ be relatively closed subsets of G with* dim $M_i=m_i<\infty$ *for* $i=1,2,\cdots,r$. *Let* $\epsilon>0$. *Then there is a collection* $\{G_i\mid i=1,2,\cdots\}$ *such that* $G=\bigcup_{i=1}^{\infty} G_i$ *and*

(i) *Each G_i is open in Y.*

(ii) $\{G_i\mid i=1,2,\cdots\}$ *is star-finite.*

(iii) $\overline{G}_i\subset G$ *for* $i=1,2,\cdots$.

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(iv) diam $G_i<\epsilon$ *for* $i=1, 2, \cdots$ *and* diam $G_i\to 0$ *as* $i\to\infty$.

(v) ord$\{G_i | G_i$ *meets* $M_1\cup M_2\cup\cdots\cup M_k\}\leqq m_1+1+m_2+1+\cdots+m_k+1$ *for* $k=1, 2, \cdots, r$.

PROOF. We sketch the proof of this proposition. From page 114 of [4] we get a collection $\{G_i' | i=1, 2, \cdots\}$ satisfying (i)–(iv). Open covers satisfying (i)–(iv) and (v) for $k=1, 2, \cdots, r$ are now defined inductively. By Proposition 1 there is a sequence $\{V_i | i=1, 2, \cdots\}$ of open sets covering M_1 such that ord$\{V_i | i=1, 2, \cdots\}\leqq m_1+1$ and $V_{k(n+1)+j}\subset G'_{k+1}$ for $k=0, 1, 2, \cdots$ and $j=1, 2, \cdots, m_1+1$. The collection $\{V_i | i=1, 2, \cdots\}\cup\{G_i'\backslash M_1 | i=1, 2, \cdots\}$ then satisfies (i)–(iv) and (v) for $k=1$.

Suppose $\{G_i' | i=1, 2, \cdots\}$ covers G, satisfies (i)–(iv) and (v) for each $k=1, 2, \cdots, n$. Let $C=M_1\cup M_2\cup\cdots\cup M_n$. By Proposition 1 there is a sequence $\{V_i | i=1, 2, \cdots\}$ of open sets covering $M_{n+1}\backslash C$ such that ord$\{V_i | i=1, 2, \cdots\}\leqq m_{n+1}+1$ and $V_{k(n+1)+j}\subset G'_{k+1}\backslash C$ for $k=0, 1, 2, \cdots$ and $j=1, 2, \cdots, m_{n+1}+1$. The collection $\{G_i'\backslash(C\cup M_{n+1}) | i=1, 2, \cdots\}\cup\{V_i | i=1, 2, \cdots\}\cup\{G_i' | G_i'$ meets $C\}$ satisfies (i)–(iv) and (v) for each $k=1, 2, \cdots, n+1$. This completes the inductive step and the sketch of the proof.

We are now in a position to prove the first theorem.

THEOREM 1. *Let C be a closed subset of a compact space Y, and let $M_1, M_2, \cdots, M_r$ be closed subsets of Y with* dim $M_i=m_i<\infty$ *for* $i=1, 2, \cdots, r$. *Let* $\epsilon>0$. *Then there is an ϵ-map $f: Y\to I^\omega$ such that $f(C)\cap f(Y\backslash C)=\emptyset$, $f|C$ is a homeomorphism, $f(Y\backslash C)$ is a countable polytope P, and* dim $f(M_i\backslash C)\leqq m_1+1+m_2+1+\cdots+m_i$ *for* $i=1, 2, \cdots, r$. *Further, $P=\Sigma_1\cup\Sigma_2\cup\cdots$ where each Σ_i is a simplex and* diam $\Sigma_i\to 0$ *as* $i\to\infty$.

PROOF. We may assume that $Y\subset I^\omega$ and that the first coordinate of each point of Y is zero. Let $\mathcal{G}=\{G_i | i=1, 2, \cdots\}$ be the open cover of $Y\backslash C$ given by Proposition 2 with diam $G_i<\epsilon/8$ for $i=1, 2, \cdots$. For each i such that $G_i\neq\emptyset$ pick a point $g_i\in G_i$. Then pick points p_i with first coordinate greater than zero such that $d(p_i, g_i)<\min\{1/i, \epsilon/8\}$ and such that $\{p_i | i=1, 2, \cdots\}$ is in general position. Let N be the collection of simplexes spanned by finite subsets $\{p_{i_0}, p_{i_1}, \cdots, p_{i_n}\}$ where $G_{i_0}\cap G_{i_1}\cap\cdots\cap G_{i_n}\neq\emptyset$. The points $\{p_i | i=1, 2, \cdots\}$ may be picked in such a way that N is a CW-polytope, and certainly $N\cap Y=\emptyset$. Also, $N=\Sigma_1\cup\Sigma_2\cup\cdots$ where each Σ_i is a simplex and diam $\Sigma_i\to 0$ as $i\to\infty$. Define f': $Y\to I^\omega$ by

$$f'(x) = \begin{cases} x & \text{if } x \in C, \\ \dfrac{\sum_{i=1}^{\infty} d(x, Y\backslash G_i)p_i}{\sum_{i=1}^{\infty} d(x, Y\backslash G_i)} & \text{if } x \notin C. \end{cases}$$

It is not hard to show that f' is continuous, and that $d(z, f'(z)) < \epsilon/4$ for each $z \in Y$. Triangulate N into simplexes of diameter less than $\epsilon/4$. By a suitable induction, a map $f_1\colon f'(Y) \cap N \to N$ may be defined in such a way that $f'(y)$ and $f_1 f'(y)$ are in the same simplexes and $f_1(f'(Y) \cap N)$ is a subpolytope P of N. The map $f\colon Y \to I^\omega$ defined by

$$f(z) = \begin{cases} z & z \in C, \\ f_1 f'(z) & z \in Y\backslash C \end{cases}$$

is then an ϵ-map such that $f(C) \cap f(Y\backslash C) = \emptyset$, $f|C$ is a homeomorphism, and $f(Y\backslash C)$ is the desired polytope P. Finally, let $y \in M_i\backslash C$. By the conditions on the cover $\mathcal{G}$, y is in at most $m_1+1+m_2+1+\cdots+m_i+1$ elements of $\mathcal{G}$. Thus $f'(y)$, and hence also $f_1f'(y)$, is in a simplex of dimension not greater than $m_1+1+m_2+1+\cdots+m_i$. Since P is a countable polytope, $\dim f(M_i\backslash C) \leqq m_1+1+m_2+1+\cdots+m_i$. Q.E.D.

Theorem 1 now enables us to prove our main theorem.

THEOREM 2. *Let X be a strongly countable dimensional G_δ space. Then there is a strongly countable dimensional compactification dX of X such that $dX\backslash X$ is a pseudo-polytope.*

PROOF. Let $X = F_1 \cup F_2 \cup \cdots$ where F_i is closed and $\dim F_i = m_i < \infty$ for $i = 1, 2, \cdots$. By a result of Hurewicz [1] there is a compactification cX of X such that $\dim \overline{F}_i^{cX} = m_i$ for $i = 1, 2, \cdots$. Let $n_i = m_1+1+m_2+1+\cdots+m_i$. Since X is a G_δ space, $cX\backslash X = Y_1 \cup Y_2 \cup \cdots$ where each Y_i is compact and $Y_i \subset Y_{i+1}$ for $i = 1, 2, \cdots$. Let $Y_0 = \emptyset$. By Theorem 1 there is a $1/i$-map $f_i\colon Y_i \to I^\omega$ such that $f_i(Y_{i-1}) \cap f_i(Y_i\backslash Y_{i-1}) = \emptyset$, $f_i|Y_{i-1}$ is a homeomorphism, $f_i(Y_i\backslash Y_{i-1})$ is a countable polytope P, and $\dim f_i(\overline{F}_k^{cX} \cap (Y_i\backslash Y_{i-1})) \leqq n_k$ for $k = 1, 2, \cdots, i$.

Decompose cX into sets $f_i^{-1}(z)$ for $z \in f_i(Y_i\backslash Y_{i-1})$ and into individual points $x \in X$. Let the quotient space be dX and let $f\colon cX \to dX$ be the quotient map. It may be shown that the decomposition of cX is upper semicontinuous, so that f is a closed map. Hence dX is a compactification of X. Further, it is easily shown that there is a uniformly continuous homeomorphism $g_i: f_i(Y_i \setminus Y_{i-1}) \to f(Y_i \setminus Y_{i-1})$. Since $f_i(Y_i \setminus Y_{i-1})$ is a countable polytope for $i=1, 2, \cdots$, $dX \setminus X$ is a pseudo-polytope.

To show that dX is strongly countable dimensional it is enough to show that $\overline{F}_i^{dX}$ is strongly countable dimensional for $i=1, 2, \cdots$. Fix a positive integer k. Since f is a closed map, $\overline{F}_k^{dX} = f(\overline{F}_k^{cX}) = F_k \cup \bigcup_{i=1}^{\infty} f(\overline{F}_k^{cX} \cap (Y_i \setminus Y_{i-1}))$. Also, $f(\overline{F}_k^{cX} \cap Y_{k-1}) \subset f(cX \setminus X) = dX \setminus X$, so $f(\overline{F}_k^{cX} \cap Y_{k-1})$ is strongly countable dimensional. Let $C_n = \bigcup_{j=k}^{n} f(\overline{F}_k^{cX} \cap (Y_j \setminus Y_{j-1}))$ for $n=k, k+1, \cdots$ and let $D_k = \bigcup_{j=k}^{\infty} C_j$. Each C_j is closed in D_k. Further, $\dim C_k = \dim f(\overline{F}_k^{cX} \cap (Y_k \setminus Y_{k-1})) \leqq n_k$. Suppose $\dim C_i \leqq n_k$. Then $C_{i+1} = C_i \cup f(\overline{F}_k^{cX} \cap (Y_{i+1} \setminus Y_i))$, C_i is closed in C_{i+1}, and $\dim f(\overline{F}_k^{cX} \cap (Y_{i+1} \setminus Y_i)) \leqq n_k$, so $\dim C_{i+1} \leqq n_k$. Therefore $\dim D_k \leqq n_k$, and $\dim D_k \cup F_k \leqq n_k + m_k + 1$. $D_k \cup F_k$ is open in $\overline{F}_k^{dX}$, so by Proposition 2 $D_k \cup F_k = \bigcup_{i=1}^{\infty} G_{ki}$, where $\overline{G}_{ki}^{dX} \subset D_k \cup F_k$ for $i=1, 2, \cdots$. Hence $\dim \overline{G}_{ki}^{dX} \leqq n_k + m_k + 1$, and $\overline{F}_k^{dX}$ is strongly countable dimensional. Q.E.D.

Sklyarenko gives an example in [9] which shows that being a G_δ space is a necessary hypothesis in Theorem 2.

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