SIMILARITY FOR SEQUENCES OF PROJECTIONS

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We consider sequences $\{P_n\}_{n=0,1,\cdots}$ of (not necessarily selfadjoint) projections in a Hilbert space H satisfying the orthogonality conditions $P_nP_m=\delta_{mn}P_n$. For brevity, such a sequence $\{P_n\}$ will be called a *p-sequence*. A p-sequence $\{E_n\}$ is *selfadjoint* if $E_n^*=E_n$ for all n. A selfadjoint p-sequence $\{E_n\}$ is *complete* if $\sum E_n$, which always converges strongly, is equal to the identity.

The object of this note is to prove the following theorem.

THEOREM. *Let $\{P_n\}$ be a p-sequence, and $\{E_n\}$ a complete selfadjoint p-sequence. Furthermore, assume that*

$$\dim P_0 = \dim E_0 = m < \infty, \tag{1}$$

$$\sum_{n=1}^{\infty} \|E_n(P_n - E_n)u\|^2 \leqq c^2\|u\|^2 \quad \textit{for all } u \in H, \tag{2}$$

where c is a constant such that $0\leqq c<1$. Then $\{P_n\}$ is similar to $\{E_n\}$, that is, there exists a nonsingular linear operator W such that

$$P_n = W^{-1}E_nW, \qquad n = 0, 1, 2, \cdots. \tag{3}$$

PROOF. First we shall show that

$$W = \sum_{n=0}^{\infty} E_nP_n \tag{4}$$

exists in the strong sense. Since $\sum E_n=1$ strongly, it suffices to show that $\sum(E_n-E_nP_n)=\sum E_n(E_n-P_n)$ converges strongly. But this is true since

$$\left\|\sum_{n=m}^{m+p} E_n(E_n - P_n)u\right\|^2 = \sum_{n=m}^{m+p} \|E_n(E_n - P_n)u\|^2 \to 0, \qquad m \to \infty, \tag{5}$$

by (2). Incidentally, we note that (5) implies $\|A\|\leqq c<1$, where

$$A = \sum_{n=1}^{\infty} E_n(E_n - P_n) = 1 - E_0 - \sum_{n=1}^{\infty} E_nP_n. \tag{6}$$

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[1] This work represents part of the results obtained while the author held a Miller Research Professorship.

Now (4) implies that $WP_n = E_nP_n = E_nW$, $n = 0, 1, 2, \cdots$. Thus the theorem will be proved if we show that W is nonsingular. To this end we consider

$$W_1 = \sum_{n=1}^{\infty} E_nP_n = 1 - E_0 - A. \tag{7}$$

Since E_0 is a selfadjoint projection with $\dim E_0 = m < \infty$, $1 - E_0$ is a *Fredholm operator* with

$$\text{nul}(1 - E_0) = m, \qquad \text{ind}(1 - E_0) = 0, \qquad \gamma(1 - E_0) = 1,$$

where nul T denotes the nullity, ind T the index, and $\gamma(T)$ the reduced minimum modulus, of the operator T (for these notions see, e.g., [2, Chapter IV, §5.1]). Since $\|A\| < 1 = \gamma(1 - E_0)$, it follows that $W_1 = 1 - E_0 - A$ is also Fredholm, with

$$\text{nul}\, W_1 \leqq \text{nul}\,(1 - E_0) = m, \qquad \text{ind}\, W_1 = \text{ind}\,(1 - E_0) = 0 \tag{8}$$

(see [2, Theorem 5.22]). Since

$$W = E_0P_0 + W_1, \tag{9}$$

where E_0P_0 is compact, W is also Fredholm and ind $W =$ ind $W_1 = 0$ (see [2, Theorem 5.26]). To show that W is nonsingular, it is therefore sufficient to show that nul $W = 0$.

To this end we first prove that

$$N(W_1) = P_0H, \tag{10}$$

where $N(T)$ denotes the null space of T. In fact, we have $W_1P_0 = 0$ by (7) so that $N(W_1) \supset P_0H$. But since $\dim P_0 = m$ and nul $W_1 \leqq m$ by (8), we must have (10).

Suppose now that $Wu = 0$. Then $0 = E_0Wu = E_0P_0u$ and $W_1u = Wu - E_0P_0u = 0$. Hence $u = P_0u$ by (10) and so $E_0u = E_0P_0u = 0$. Thus $(1 - A)u = (W_1 + E_0)u = 0$ by (7). Since $\|A\| < 1$, we obtain $u = 0$. This shows that nul $W = 0$ and completes the proof.

REMARK. It has been shown by C. Clark [1] that the theorem is useful in proving that certain ordinary differential operators are spectral in the sense of Dunford.

BIBLIOGRAPHY

1. C. Clark, *On relatively bounded perturbations of ordinary differential operators*, Pacific J. Math. (to appear)

2. T. Kato, *Perturbation theory for linear operators*, Springer, Berlin, 1966.

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