DEMICONTINUITY, HEMICONTINUITY AND MONOTONICITY. II

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In the previous paper [6] with the same title, the writer proved that a (nonlinear) monotonic operator G from a Banach space X to the adjoint space X^* is demicontinuous if and only if it is hemicontinuous and locally bounded, under a certain mild assumption on $D(G)$. (For similar results see also Browder [3].) In the present note we shall show that if $D(G)$ is an open set, the assumption of local boundedness is redundant so that hemicontinuity and demicontinuity are equivalent. Furthermore, we shall show that a similar result holds for a more general class of operators, including *accretive operators* in X where X^* is uniformly convex.

In what follows we consider (real or complex) Banach spaces X, Y and (nonlinear) operators F, G such that (D and R denoting the domain and range, respectively) $D(F)=X$, $R(F)\subset Y$, $D(G)\subset X$, $R(G)\subset Y^*$.

DEFINITION 1. G is said to be F-monotonic if

$$\operatorname{Re}(F(x-y), Gx-Gy) \geqq 0, \qquad x, y \in D(G),$$

where (,) denotes the pairing between Y and Y^*.

DEFINITION 2. Let $u\in D(G)$. G is said to be

(a) demicontinuous at u if $u_n\in D(G)$, $n=1, 2, \cdots$, and $u_n\to u$ as $n\to\infty$ imply $Gu_n\rightharpoonup Gu$ (here and in what follows $\to$ and $\rightharpoonup$ denote strong and weak* convergence, respectively);

(b) locally bounded at u if the conditions in (a) imply that $\|Gu_n\|$ is bounded as $n\to\infty$;

(c) hemicontinuous at u if $v\in X$, $t_n>0$, $n=1, 2, \cdots$, $t_n\to 0$ and $u+t_nv\in D(G)$ imply $G(u+t_nv)\rightharpoonup Gu$;

(d) locally hemibounded at u if the conditions in (c) imply that $\|G(u+t_nv)\|$ is bounded as $n\to\infty$.

Obviously we have the implications

$$(a) \begin{array}{c} \nearrow\!\!\!\!\nearrow \\ \searrow\!\!\!\!\searrow \end{array} \begin{array}{c} (b) \\ (c) \end{array} \begin{array}{c} \searrow\!\!\!\!\searrow \\ \nearrow\!\!\!\!\nearrow \end{array} (d).$$

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EXAMPLE 1. Let $Y=X$ and $F=1$ (the identity operator in X). Then the F-monotonicity of G means that G is monotonic in the usual sense of Minty [8] and Browder [2].

EXAMPLE 2. Assume that X is reflexive and set $Y=X^*$. Further assume that X^* is strictly convex. Let $F: X\to X^*$ be the duality map, which is well defined if we assume that $\|Fx\|=\|x\|$ (see Beurling and Livingston [1], Browder [4]). In this case the F-monotonicity of G (which has range space $Y^*=X$) means that G is accretive in the sense of Browder [5]. We note that here F is itself a monotonic operator in the sense of Example 1. Since F is also known to be demicontinuous and coercive (see Browder [5]), it follows from a theorem of [8] and [2] that F is onto X^*. We note also that F is uniformly continuous (in the strong topologies) in any bounded subset of X if X^* is uniformly convex (see e.g. Kato [7]).

THEOREM. *Assume that*

(1°) *F is positive-homogeneous*: $F(tx)=tF(x)$ *for* $t>0$;

(2°) *F is onto Y*: $R(F)=Y$;

(3°) *F is uniformly continuous in the closed unit ball of X (in the strong topologies)*;

(4°) *$D(G)$ is open in X*;

(5°) *G is F-monotonic.*

Then

(i) *G is locally bounded at $u\in D(G)$ if and only if it is locally hemibounded at u*;

(ii) *G is demicontinuous at $u\in D(G)$ if and only if it is hemicontinuous at u.*

In view of Examples 1, 2 given above, the following corollary is a direct consequence of the theorem.

COROLLARY. *Demicontinuity and hemicontinuity are equivalent for G in each of the following cases.*

(a) *G is a monotonic operator with $D(G)$ open in X and $R(G)\subset X^*$, X being an arbitrary Banach space.*

(b) *G is an accretive operator in X with $D(G)$ open, X^* being assumed to be uniformly convex.*

PROOF OF THE THEOREM. (i) Since local boundedness at u implies local hemiboundedness at u, it suffices to prove the converse implication. To this end, it suffices to show that $u_n\to u$ and $\|Gu_n\|=r_n\to\infty$ lead to a contradiction if G is locally hemibounded at $u\in D(G)$.

For each $s>0$ let $\phi(s)$ be the supremum of $\|Fx-Fy\|$ for $\|x\|\leqq 1$, $\|y\|\leqq 1$, and $\|x-y\|\leqq s$. Since by (3°) F is uniformly continuous in the unit ball of X, $\phi(s)$ is nondecreasing in s and $\phi(s)\to 0$ as $s\to 0$. Furthermore, $\phi(s)<\infty$ for all $s>0$ in virtue of positive-homogeneity (1°). Thus

$$(1)\qquad \|Fx - Fy\| \leqq \phi(\|x-y\|) \quad \text{if} \quad \|x\|\leqq 1,\ \|y\|\leqq 1.$$

Set

$$(2)\qquad t_n = \operatorname{Max}[(1/r_n),\ \|u_n-u\|^{1/2},\ \phi(\|u_n-u\|)^{1/2}],$$

so that

$$(3)\qquad t_n>0,\quad t_nr_n\geqq 1,\quad \|u_n-u\|\leqq t_n^2,\quad \phi(\|u_n-u\|)\leqq t_n^2.$$

Since $r_n\to\infty$ and $\|u_n-u\|\to 0$ by hypothesis, $\phi(\|u_n-u\|)\to 0$ too and

$$(4)\qquad t_n\to 0 \quad \text{as} \quad n\to\infty.$$

Let $v\in X$ and $w_n=u+t_nv$. Since $u\in D(G)$ and $D(G)$ is open, $w_n\in D(G)$ for sufficiently large n. Thus we have by (1°) and (5°)

$$(5)\qquad \begin{aligned}\operatorname{Re}(Fv, Gu_n) \leqq{}& t_n^{-1}\operatorname{Re}(F(w_n-u_n), Gw_n)\\ &+ t_n^{-1}\operatorname{Re}(F(t_nv)-F(w_n-u_n), Gu_n).\end{aligned}$$

Let us estimate the right member of (5). First we note that $t_n^{-1}(w_n-u_n)=v-t_n^{-1}(u_n-u)\to v$ as $n\to\infty$ for $t_n^{-1}\|u_n-u\|\leqq t_n^{-1}\to 0$ by (3) and (4). Since F is continuous by (1°) and (3°), we have $t_n^{-1}F(w_n-u_n) = F(t_n^{-1}(w_n-u_n))\to Fv$. Since $\|Gw_n\|=\|G(u+t_nv)\|$ is bounded as $n\to\infty$ by the assumed local hemiboundedness at u of G, the first term on the right of (5) is thus bounded as $n\to\infty$.

To estimate the second term, we note that both t_nv and w_n-u_n tend to zero and hence belong to the unit ball of X for sufficiently large n. Hence $\|F(t_nv)-F(w_n-u_n)\|\leqq\phi(\|t_nv-w_n+u_n\|)=\phi(\|u_n-u\|)\leqq t_n^2$ by (1) and (3), so that the second term on the right of (5) is majorized by $t_n\|Gu_n\|=t_nr_n$.

Thus we have the estimate

$$(6)\qquad \operatorname{Re}(Fv, Gu_n)\leqq C+t_nr_n,$$

where C may depend on v but not on n. Dividing (6) by t_nr_n and noting that $t_nr_n\geqq 1$ by (3), we obtain

$$(7)\qquad \limsup_{n\to\infty}\operatorname{Re}(Fv, (t_nr_n)^{-1}Gu_n)<\infty.$$

Since F is onto Y by (2°), $y=Fv$ in (7) can be any element of Y. Replacing y by $-y$ (and by $\pm iy$ if Y is complex), we see that $(y, (t_nr_n)^{-1}Gu_n)$ is bounded as $n\to\infty$ for every $y\in Y$. But this is a contradiction to the principle of uniform boundedness, for $\|(t_n r_n)^{-1} G u_n\| = t_n^{-1} \to \infty$.

(ii) Again it suffices to show that G is demicontinuous at $u \in D(G)$ if it is hemicontinuous at u. Let $u_n \to u$; we have to show that $Gu_n \rightharpoonup Gu$. Since hemicontinuity at u implies local hemiboundedness at u, G is locally bounded at u by (i) just proved. Thus $r_n = \|Gu_n\|$ is bounded as $n \to \infty$.

Now we set

$$t_n = \operatorname{Max}[\|u_n - u\|^{1/2}, \phi(\|u_n - u\|)^{1/2}],$$

so that (3) and (4) are again true except $t_n r_n \geqq 1$. With $w_n = u + t_n v$, we have again the inequality (5). Since $t_n^{-1} F(w_n - u_n) \to Fv$ as before and since $Gw_n = G(u + t_n v) \rightharpoonup Gu$ by the hemicontinuity of G at u, the first term on the right of (5) tends to $\operatorname{Re}(Fv, Gu)$. The second term is majorized by $t_n r_n$ as before. Since r_n is now bounded, this term tends to zero. Thus we have

$$\limsup_{n \to \infty} \operatorname{Re}(Fv, Gu_n - Gu) \leqq 0.$$

Since $Fv = y$ may be an arbitrary element of Y, it follows by an argument similar to the one used in (i) that $\limsup |(y, Gu_n - Gu)| = 0$ for every $y \in Y$. Thus $Gu_n \rightharpoonup Gu$.

Bibliography

1. A. Beurling and A. E. Livingston, *A theorem on duality mappings in Banach spaces*, Ark. Mat. **4** (1962), 405–411.

2. F. E. Browder, *Nonlinear elliptic boundary value problems*, Bull. Amer. Math. Soc. **69** (1963), 862–874.

3. ———, *Continuity properties of monotone nonlinear operators in Banach spaces*, Bull. Amer. Math. Soc. **70** (1964), 551–553.

4. ———, *On a theorem of Beurling and Livingston*, Canad. J. Math. **17** (1965), 367–372.

5. ———, *Nonlinear accretive operators in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 470–476.

6. T. Kato, *Demicontinuity, hemicontinuity and monotonicity*, Bull. Amer. Math. Soc. **70** (1964), 548–550.

7. ———, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan. (to appear).

8. G. J. Minty, On a *"monotonicity" method for the solution of nonlinear equations in Banach spaces*, Proc. Nat. Acad. Sci. **50** (1963), 1038–1041.

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