STRONG HOMOTOPY EQUIVALENCE OF 3-MANIFOLDS

BY D. R. McMILLAN, JR.[1]

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1. **Introduction.** Let M be a topological space and let X be a compact subset of M. After [2], we say that X has property UV^∞ (or "$X \in UV^\infty$") in M if for each open set $U \subset M$ such that $X \subset U$, there is an open set V such that $X \subset V \subset U$ and V is contractible to a point in U. It is known [2] that each finite-dimensional compact absolute retract (i.e., retract of a cell) has property UV^∞ under some embedding in Euclidean space, and that if one embedding of a compact set X in a manifold has property UV^∞, then so does every embedding of X in a manifold.

Armentrout has shown (§10 of [2]) that if M^n and N^n are *closed* (i.e., compact and boundaryless) topological n-manifolds, and if f is a continuous mapping of M^n onto N^n such that $f^{-1}(y) \in UV^\infty$ for each $y \in N^n$, then f is a homotopy equivalence. We shall call such a mapping a *strong homotopy equivalence of* M^n onto N^n. If $n=3$ and if f is *cellular* (i.e., each set $f^{-1}(y)$ is cellular—and hence UV^∞), then he has shown [1] that M^3 and N^3 are homeomorphic. It is our purpose here to note that if there is a strong homotopy equivalence of M^3 onto N^3, then M^3 and N^3 differ by only a finite number of homotopy 3-cells (Corollary 2.1). Hence, modulo the Poincaré conjecture, M^3 and N^3 are homeomorphic. If there is also a strong homotopy equivalence of N^3 onto M^3, then M^3 and N^3 *are* homeomorphic (independently of the Poincaré conjecture).

If X is a compact subset of the interior of a piecewise-linear n-manifold M^n, $n \geqq 3$, and if $X \in UV^\infty$, then we shall say that X satisfies the *cellularity criterion* in M^n if for each open set $U \subset M^n$ such that $X \subset U$, there is an open set V such that $X \subset V \subset U$ and each loop in $V-X$ is contractible in $U-X$. If $n \geqq 5$ and $X \in UV^\infty$, then X is cellular (with respect to piecewise-linear cells) in M^n if and only if the cellularity criterion holds (see [5]). For the situation in the 3-dimensional case, see Theorem 1.

We shall use E^n and S^n to denote Euclidean n-space, and the n-sphere, respectively. The term "manifold" applies only to a connected space, unless stated otherwise. If G is a disjoint collection of closed subsets of a space X such that the union of the elements of G is

[1] Alfred P. Sloan Fellow.

X, we shall say that G is an *upper semicontinuous decomposition of* X if for each $g \in G$ and for each open set $U \subset X$ such that $g \subset U$, there is an open set V such that $g \subset V \subset U$ and such that each element of G which intersects V is contained in U.

2. **Cellularity of inverse sets.** The following is an improved version of Theorem 1′ of [5].

THEOREM 1. *Let M^3 be a piecewise-linear 3-manifold without boundary and let X be a compact subset of M^3 such that $X \in UV^\infty$ and X satisfies the cellularity criterion in M^3. Then, for each open set $U \subset M^3$ such that $X \subset U$, there is a compact, polyhedral, contractible 3-manifold H such that*

$$X \subset \operatorname{Int} H \subset H \subset U,$$

and such that $H - X$ is topologically $S^2 \times [0, 1)$.

PROOF. We assume a familiarity with Lemma 1 of [5]. Let an arbitrary open set $V \subset M^3$ be given such that $X \subset V$. The first step is to show that there is a compact 3-manifold-with-boundary $H \subset V$ such that $X \subset \operatorname{Int} H$ and such that H is a "homotopy cube-with-handles," that is, H is obtained from a homotopy 3-cell by adding orientable handles of index one to its boundary.

To do this, we make several applications of the UV^∞ property and of regular neighborhoods to find compact 3-manifolds-with-boundary K and M_1 (K may not be connected) such that

$$X \subset \operatorname{Int} K \subset K \subset \operatorname{Int} M_1 \subset M_1 \subset V,$$

such that M_1 is contractible in V and Bd M_1 is connected and nonempty, and such that each loop in K is contractible in M_1. Note that each polyhedral 2-sphere in M_1 bounds a homotopy 3-cell in V and, since Bd M_1 is connected, this homotopy 3-cell lies in M_1. We are now in a position to repeat, using the same notation, the argument outlining the proof of Lemma 1 in [5]. The only difference is that in the present situation we know only that a polyhedral 2-sphere in M_1 bounds a homotopy 3-cell in M_1, rather than a piecewise-linear 3-cell. This will complete the first step of the proof, and makes use only of the fact that $X \in UV^\infty$.

The second step is to show, using the fact that X satisfies the cellularity criterion, that H can be chosen to be a homotopy 3-cell. For this we appeal to the proof of Theorem 1′ of [5], to "cut the handles" of our homotopy 3-cell-with-handles, without deleting any part of X in the process.

By the first two steps, we may find homotopy 3-cells $H_1, H_2, \cdots$, such that $H_{i+1} \subset \operatorname{Int} H_i$ and $X = \bigcap_{i=1}^{\infty} H_i$. The third step, which will complete the proof, is to show that there exists an integer N such that $A_i = H_i - \operatorname{Int} H_{i+1}$ is topologically $S^2 \times [0, 1]$ (a "3-annulus") for each $i > N$.

Let $F_i \subset \operatorname{Int} A_i$ be a polyhedral homotopy 3-cell obtained from A_i by tunneling along an arc from one component of Bd A_i to the other, and then shaving off a product neighborhood of the boundary of the resulting 3-manifold. Then F_i is a 3-cell if and only if A_i is a 3-annulus. A result of Kneser (pp. 252–255 of [**3**]) implies that, since H_1 is compact, there is an integer N such that H_1 does not contain more than N disjoint, polyhedral homotopy 3-cells which fail to be 3-cells (apply Kneser's result to the "double" of H_1). This is the required N. The theorem follows.

COROLLARY 1.1. *Assume the hypotheses of Theorem* 1, *and let G be the upper semicontinuous decomposition of M^3 whose only nondegenerate element is X. Then the decomposition space M^3/G is a* 3-*manifold which can be obtained from M^3 by removing a compact, polyhedral homotopy* 3-*cell and replacing it by a piecewise-linear* 3-*cell.*

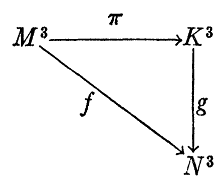
THEOREM 2. *Suppose that M^3 and N^3 are closed piecewise-linear* 3-*manifolds and that f is a continuous mapping of M^3 onto N^3 such that $f^{-1}(y) \in UV^{\infty}$, for each $y \in N^3$. Then, for each $y \in N^3$, and for each open set $U \subset M^3$ containing $X = f^{-1}(y)$, there is an open set V such that $X \subset V \subset U$ and $V - X$ is topologically $S^2 \times (0, 1)$.*

PROOF. Using the techniques of [**7**, Theorem 2.1], or [**4**, Lemma 5], or [**2**, Corollary 6.5], we see that each set $f^{-1}(y)$ satisfies the cellularity criterion in M^3. The result follows by Theorem 1.

COROLLARY 2.1. *Under the hypotheses of Theorem* 2, *there are at most a finite number of $y \in N^3$ such that $f^{-1}(y)$ fails to be cellular in M^3. Further, N^3 can be obtained by removing a finite, disjoint collection of compact, polyhedral homotopy* 3-*cells from M^3 and replacing each by a piecewise-linear* 3-*cell.*

PROOF. The fact that only a finite number of inverse sets can fail to be cellular is immediate from Theorem 2, the compactness of M^3, and the fact that $\{f^{-1}(y) \mid y \in N^3\}$ is an upper semicontinuous decomposition of M^3 into compact sets. To prove the second assertion, let $y_1, \cdots, y_n$ be all those points y of N^3 for which $f^{-1}(y)$ fails to be cellular. Let π be a mapping of M^3 onto a closed 3-manifold K^3 such that the only nondegenerate inverse sets of π are $f^{-1}(y_1), \cdots, f^{-1}(y_n)$, and such that K^3 is (by Corollary 1.1) obtained from M^3 by a finite number of "surgeries" of the type described.

Let g be defined so as to make the following diagram consistent:



Then g is a cellular mapping, and by [1], N^3 is homeomorphic to K^3, as required.

COROLLARY 2.2 *Under the hypotheses of Theorem* 2, *if* M^3 *and* N^3 *are homeomorphic, then each set* $f^{-1}(y)$ *is cellular in* M^3.

PROOF. Reviewing the proof of Corollary 2.1, we see that it will suffice to show that if $F_1, \cdots, F_n$ is a finite disjoint collection of polyhedral, homotopy 3-cells in M^3, such that each F_i fails to be a topological 3-cell, and if each F_i is replaced by a (real) 3-cell, then the resulting 3-manifold is *not* homeomorphic to M^3. If M^3 is an orientable 3-manifold, this is immediate from Milnor's "unique decomposition theorem" (Theorem 1 of [6]). If M^3 is nonorientable, we apply the same argument to the orientable double covering of M^3.

COROLLARY 2.3. *Let* M^3 *and* N^3 *be closed, piecewise-linear* 3-*manifolds, and suppose that there exist continuous mappings*

$$f_1: M^3 \to N^3, \qquad f_2: N^3 \to M^3$$

of each of these 3-*manifolds onto the other, such that* $f_1^{-1}(y) \in UV^\infty$ *and* $f_2^{-1}(x) \in UV^\infty$, *for each* $x \in M^3, y \in N^3$. *Then* M^3 *and* N^3 *are homeomorphic, and each point inverse of* $f_i (i=1, 2)$ *is cellular.*

PROOF. This follows from Corollary 2.1 and Corollary 2.2 (and its proof).

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THE UNIVERSITY OF WISCONSIN

THE C^*-ALGEBRA GENERATED BY AN ISOMETRY[1]

BY L. A. COBURN

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1. Introduction. In this paper, I determine the structure of any C^*-algebra generated by an isometry. Using a theorem of Halmos [3], the problem is reduced to the study of C^*-algebras $\mathcal{A}(A)$ generated by A and A^* where (i) A is unitary, (ii) $A=S_\alpha$ with S_α the shift of multiplicity α, and (iii) $A=W\oplus S_\alpha$ with W unitary. In case (i), the resulting algebra is isometrically *-isomorphic to the algebra $C(\sigma(A))$ of all complex-valued continuous functions on the spectrum of A and nothing more need be said. In cases (ii) and (iii), it turns out that $\mathcal{A}(A)$ is isometrically *-isomorphic to $\mathcal{A}(S_1)$ so that $\mathcal{A}(A)$ is independent of W and α. In each of these cases, there is a unique minimal closed two-sided ideal $\mathcal{J}(A)$ such that $\mathcal{A}(A)/\mathcal{J}(A)$ is isometrically *-isomorphic to $C(T)$, where T is the perimeter of the unit circle. The ideal $\mathcal{J}(A)$ is determined spatially in the cases $A=S_1$ and $A=W\oplus S_1$.

We begin with the notation. For our purposes, all Hilbert spaces are complex and all ideals are closed and two-sided. If $\{e_n: n=0, 1, 2, \cdots\}$ is an orthonormal basis for a Hilbert space H then the shift $S=S_1$ is defined by $Se_n=e_{n+1}$. By a shift of multiplicity α is meant the α-fold direct sum $S\oplus S\oplus \cdots \oplus S$ acting on $H\oplus H\oplus \cdots \oplus H$. The orthogonal projection onto the one-dimensional subspace of H spanned by e_n is denoted by P_n.

If H (or H_i) is a Hilbert space then $\mathcal{B}(H)$ (or $\mathcal{B}(H_i)$) denotes the algebra of all bounded operators with the usual norm topology and $\mathcal{K}$ (or $\mathcal{K}_i$) denotes the ideal of all compact operators. The natural quotient map from $\mathcal{B}(H)$ to $\mathcal{B}(H)/\mathcal{K}$ ($\mathcal{B}(H_i)$ to $\mathcal{B}(H_i)/\mathcal{K}_i$) is given by

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