# SOME REMARKS ON PARALLELIZABLE STEIN MANIFOLDS

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Communicated by John Wermer, April 3, 1967

1. The purpose of this note is to collect some simple facts on the parallelizability of analytic submanifolds of the complex number space $C^N$, which are remarkable because their analogues in the real case fail to be true. Any analytic submanifold of $C^N$ is a Stein manifold. (An analytic submanifold is closed in $C^N$ by definition.) Conversely, every Stein manifold can be embedded in some $C^N$, i.e. mapped biholomorphically onto an analytic submanifold of $C^N$. An $n$-dimensional Stein manifold $X$ is called parallelizable if there exists a holomorphic field of $n$-frames on $X$, i.e. $n$ holomorphic vector fields which are linearly independent at every point $x \in X$. (We require throughout this paper that all connected components of a manifold have the same dimension.) By a theorem of Grauert [2], an $n$-dimensional Stein manifold $X$ is parallelizable if and only if there exists a continuous field of (complex) $n$-frames on $X$. We connect the parallelizability with the notion of complete intersection: An $n$-dimensional analytic submanifold $X$ of $C^N$ is called a complete intersection, if the ideal $I(X)$ of all holomorphic functions on $C^N$ which vanish on $X$ can be generated by $N-n$ elements. This is the case if and only if there exist $N-n$ holomorphic functions $f_1, \cdots, f_{N-n}$ on $C^N$ such that

$$X = \{x \in C^N : f_1(x) = \cdots = f_{N-n}(x) = 0\}$$

and the rank of the functional matrix of $(f_1, \cdots, f_{N-n})$ equals $N-n$ at every point $x \in X$. We shall prove that a Stein manifold is parallelizable if and only if it can be embedded as a complete intersection in some complex number space $C^N$.

2. The following lemma expresses the duality between the normal and tangent bundle of an analytic submanifold of $C^N$.

LEMMA. *Let $X$ be an $n$-dimensional analytic submanifold of $C^N$.*

(i) *If the normal bundle of $X$ is trivial, then $X$ is parallelizable.*

(ii) *If $X$ is parallelizable and $N \geqq 3n/2$, then the normal bundle of $X$ is trivial.*

[1] Supported by National Science Foundation grant GP-5803.

PROOF. We use the following theorem (cf. [3]). Let $\xi_1$ and $\xi_2$ be two complex vector bundles of rank $r$ over an $n$-dimensional CW-complex $X$ and $\eta$ a trivial vector bundle over $X$. Suppose that $\xi_1\oplus\eta$ is isomorphic to $\xi_2\oplus\eta$ and that $r\geqq n/2$. Then $\xi_1$ and $\xi_2$ are isomorphic.

Let $\tau$ be the tangent bundle of $X$, $\nu$ the normal bundle of $X$ and $\theta$ the (trivial) tangent bundle of $C^N$, restricted to $X$. Then we have $\tau\oplus\nu\cong\theta$. In order to prove (i) we apply the cited theorem to the case where $\eta=\nu$, $\xi_1=\tau$ and $\xi_2$ is the trivial vector bundle of rank $n$ over $X$. In order to prove (ii) we set $\eta=\tau$, $\xi_1=\nu$ and $\xi_2=$ trivial vector bundle of rank $N-n$ over $X$, using the fact that a (complex) $n$-dimensional Stein manifold has the homotopy type of a (real) $n$-dimensional CW-complex (cf. [5]).

THEOREM. *Let $X$ be an $n$-dimensional analytic submanifold of $C^N$.*

(i) *If $X$ is a complete intersection, then it is parallelizable.*

(ii) *If $X$ is parallelizable and $N\geqq(3n/2)+1$, then $X$ is a complete intersection.*

PROOF. (i) Because the normal bundle of a complete intersection is trivial, this follows from part (i) of the lemma.

(ii) Part (ii) of the lemma implies that the normal bundle of $X$ is trivial. Since $N\geqq(3n/2)+1$, it follows from [1, Satz 12], that $X$ is a complete intersection.

COROLLARY 1. *A Stein manifold is parallelizable if and only if it can be embedded as a complete intersection in some complex number space $C^N$.*

This is true because every $n$-dimensional Stein manifold can be embedded in $C^{2n+1}$.

COROLLARY 2. *Every $(N-1)$-dimensional analytic submanifold of $C^N$ is parallelizable.*

Since the second Cousin problem in $C^N$ always has a solution, every such submanifold is a complete intersection.

Note that the analogue of the corollary for the real case is not true as the two-sphere

$$\{(x, y, z) \in R^3 : x^2 + y^2 + z^2 = 1\}$$

shows. (However it is well known that the complexified two-sphere

$$\{(x, y, z) \in C^3 : x^2 + y^2 + z^2 = 1\}$$

is parallelizable.)

Corollary 3. *An $(N-2)$-dimensional analytic submanifold $X$ of $C^N$ is parallelizable if and only if its first Chern class $c_1(X)$ vanishes.*

Proof. It is clear that all Chern classes of $X$ vanish, if $X$ is parallelizable. Conversely, suppose $c_1(X)=0$. Then it has been proved in [1, Hilfssatz 17, Corollary], that $X$ is a complete intersection. Therefore the corollary follows from part (i) of the theorem.

Using this last argument and part (ii) of the theorem, we get

Corollary 4. *An analytic submanifold $X$ of $C^N$, $N\leqq 7$, is a complete intersection if and only if $X$ is parallelizable.*

Thus the property of being a complete intersection in $C^N$, $N\leqq 7$, depends only on the submanifold and not on its embedding in $C^N$. It is not known if this remains true in $C^N$ with $N\geqq 8$.

A theorem of Peterson [6] (cf. also [4]) asserts that if $X$ is an $n$-dimensional CW-complex such that the only torsion in $H^{2q}(X, Z)$ is relatively prime to $(q-1)!$ for $q=1, 2, \cdots$, and $\xi$ is a complex vector bundle of rank $\geqq n/2$ over $X$, then $\xi$ is trivial if and only if all its Chern classes vanish. Since for a (complex) $n$-dimensional Stein manifold $X$ we have $H^p(X, Z)=0$ for $p>n$, we can express the parallelizability of low-dimensional Stein manifolds completely in terms of Chern classes.

Proposition. *A Stein manifold $X$ of dimension $\leqq 5$ is parallelizable if and only if $c_1(X)=c_2(X)=0$.*

3. We want to give an example of a Stein manifold which is not parallelizable. Since every one-dimensional Stein manifold is parallelizable (this follows from the above proposition, but can also be proved by more elementary means), the simplest example is of dimension two. Let $X$ be the following open subset of the 2-dimensional complex projective space:

$$X = \{(x:y:z) \in P_2(C): x^2+y^2+z^2 \neq 0\}.$$

It is easy to check that $X$ is Stein. We claim that $X$ is not parallelizable. The real projective plane $P_2(R)$, which is naturally embedded in $X$, is a deformation retract of $X$. A deformation of $X$ to $P_2(R)$ is given by the family of maps $F_t: X\rightarrow X$, $1\geqq t\geqq 0$, where

$$F_t(x:y:z) = ((\mathrm{Re}\, x + it\, \mathrm{Im}\, x):(\mathrm{Re}\, y + it\, \mathrm{Im}\, y):(\mathrm{Re}\, z + it\, \mathrm{Im}\, z)).$$

Here the homogeneous coordinates $(x:y:z)$ of a point of $X$ have to be chosen in such a way that $x^2+y^2+z^2>0$. Let $\tau$ be the tangent bundle of $X$. We want to calculate the Stiefel-Whitney classes of $\tau$ (regarded as a real vector bundle). It suffices to consider the restriction of $\tau$ to $P_2(R)$. This restriction is the complexification of $\tau_0$, the real tangent bundle of $P_2(R)$, hence, as a real bundle, isomorphic to the Whitney sum $\tau_0 \oplus \tau_0$. The total Stiefel-Whitney class of $\tau_0$ is $w(\tau_0) = (1+\alpha)^3$, where $\alpha \in H^1(P_2(R), Z_2)$ is the generator of the cohomology ring $H^*(P_2(R), Z_2)$, (cf. [3]). Therefore $w(\tau_0 \oplus \tau_0) = (1+\alpha)^6 = 1+\alpha^2$. Since the second Stiefel-Whitney class is the reduction modulo 2 of the first Chern class, it follows that $c_1(X) = c_1(\tau) = \gamma$, where $\gamma$ is the nonzero element of $H^2(X, Z) \cong H^2(P_2(R), Z) \cong Z_2$. This shows that $X$ is not parallelizable.

By Corollary 2, $X$ cannot be embedded in $C^3$. However, the mapping $f: X \to C^4$,

$$f(x:y:z) = (x^2 + y^2 + z^2)^{-1}(xy, xz, yz, x^2 - y^2),$$

maps $X$ biholomorphically onto an analytic submanifold $X_1$ of $C^4$. By our theorem, $X_1$ is not a complete intersection (cf. Stiefel [8, Anhang II]).

The Stein manifold $X$ can also be used to show that the following result of Ramspott [7] is best possible.

*Let $Y$ be an $n$-dimensional Stein manifold. Then for every $k \leqq (n+1)/2$ there exists a holomorphic field of $k$-frames on $Y$.*

For $n = 2m$ set $Y = X^m$, for $n = 2m+1$ set $Y = X^m \times C$. We claim that there exists no holomorphic field of $k$-frames on $Y$ if $k > (n+1)/2$. To prove this it suffices to show that $c_m(Y) \neq 0$. The cohomology ring $H^*(Y, Z_2) \cong H^*((P_2(R))^m, Z_2)$ is isomorphic to $Z_2[\alpha_1, \cdots, \alpha_m]$, where $\alpha_1, \cdots, \alpha_m$ are elements of degree 1 with the only relation $\alpha_i^3 = 0$. From our above calculations on $X$ follows that the total Stiefel-Whitney class of the tangent bundle $\tau$ of $Y$ is

$$w(\tau) = (1 + \alpha_1^2) \cdot \ldots \cdot (1 + \alpha_m^2).$$

In particular $w_{2m}(\tau)$ is the nonzero element of $H^{2m}(Y, Z_2) = Z_2$. Therefore $c_m(Y)$ is the nonzero element of $H^{2m}(Y, Z) = Z_2$. q.e.d.

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# AUTOMORPHISM GROUPS OF FINITELY GENERATED NILPOTENT GROUPS

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Communicated by David A. Buchsbaum, April 5, 1967

It is rare for any property of a group $G$ to carry over to its automorphism group. Recently J. Lewin [1] constructed a finitely presented group whose automorphism group is not even finitely generated. Now finitely generated nilpotent groups are finitely presented (see e.g. [2]). So Lewin's example contrasts strikingly with the following.

THEOREM A. *The automorphism group of a finitely generated nilpotent group is finitely presented.*

In a way Theorem A reinforces the commonly held view that the automorphism group of a finitely generated nilpotent group is, from a group-theoretical viewpoint, quite simple. Now Philip Hall [3] has proved that a finitely generated nilpotent group has a faithful representation in $\mathrm{GL}(n, Z)$, the integer unimodular group of degree $n$. So the following generalization of Hall's theorem might be thought of as another indication of the controlled nature of finitely generated nilpotent groups and their automorphism groups.

THEOREM B. *The holomorph of a finitely generated nilpotent group (i.e. the split extension of the group by its automorphism group) has a faithful representation in* $\mathrm{GL}(n, Z)$ *for some* $n$.

The proofs of Theorem A and Theorem B use general Lie-theoretic techniques and a result which is of independent interest, namely

[1] Help from the N.S.F. is gratefully acknowledged. The second author holds a Sloan Fellowship.